Release date: June 23, 2020 Deadline: June 30, 2020: 23:00

Refer to recorded lectures on the prophet and the prophet-secretary problems for this homework. Submission format: PDF is preferred. Photographs and scans are also acceptable, but it's your responsibility to ensure that they are clear and readable.

In this homework we will analyze a fixed-threshold algorithm for the prophet-secretary problem, for the case when the CDFs $F_1, \ldots, F_n$ of the independent random variables $X_1, \ldots, X_n$ are all continuous, that is, none of the probability distributions have point masses. Observe that this implies that there exists a τ such that $\Pr[\max_i X_i \le \tau] = \prod_{i=1}^{n} F_i(\tau) = 1/e$. We will analyze the algorithm that uses this τ as the fixed threshold, that is, it accepts the earliest value that exceeds τ. Like in the recorded lectures, we write the algorithm's reward as a sum of revenue and utility.

1. **[1 point]** Determine the expected revenue of the algorithm.
2. To analyze the expected utility, it is convenient to imagine that each random variable X_i appears at a uniformly random arrival time t_i in $[0, 1]$, and these n arrival times are independent (like in the real-time prophet-secretary problem defined in the recorded lectures). Let the random variable T denote the stopping time of the algorithm. Like in the prophet secretary analysis, we define $\theta(t) = \Pr[T \ge t]$, the probability that the algorithm doesn't stop before time t.
   (a) **[4 points]** Show that $\theta(t) = \prod_{i=1}^{n}(1 - t + t \cdot F_i(\tau))$. Hence, prove that $\theta(t) \ge e^{-t}$. (Hint: AM-GM inequality.)
   (b) **[4 points]** Observe that for all i, we have $\theta(t) \le \Pr[T \ge t \mid t_i \ge t] = \Pr[T \ge t \mid t_i = t]$. (You don't have to write the proof of this; it is the same as in the recorded lecture.) Using this fact and the bound on $\theta(t)$ you just proved, show that the expected utility is bounded from below by $(1 - 1/e) \cdot \mathbb{E}[(\max_i X_i - \tau)^+]$. (As usual, a^+ denotes $\max(a, 0)$.)
3. **[1 point]** Using the bounds on the expected revenue and the expected utility, derive a competitiveness guarantee of the algorithm.

Note: the idea can be extended to handle the case when F_i's have point masses. Figure out the details if you are interested.