COL 726

Homework IV

Due on April 1, 2018

1. For an arbitrary $m \times m$ matrix A with complex entries, let $\rho(A)$ denote the largest absolute value of an eigenvalue of A, and $\alpha(A)$ denote the maximum over all eigenvalues λ of A of the real part of λ. Prove the following :

   (a) $\lim_{n\to\infty} ||A^n|| = 0$ if and only if $\rho(A) < 1$.

   (b) $\lim_{t\to\infty} ||e^{tA}|| = 0$ if and only if $\alpha(A) < 0$.

2. Suppose A is a real symmetric matrix with one eigenvalue much smaller than the rest in absolute value. Suppose we run the inverse power iteration algorithm with $\mu = 0$. Let $q_1, \ldots, q_n$ be n orthonormal eigenvectors of A. Suppose v is a vector which has non-zero components along each of these eigenvectors, and suppose we solve for $Aw = v$ (as in the inverse power iteration algorithm). Suppose we compute a vector $\tilde{w}$ here (using a backward stable algorithm). Show that the vectors $w/||w||$ and $\tilde{w}/||\tilde{w}||$ are close to each other.

3. Let A be a tridiagonal Hermitian matrix with all its sub- and super-diagonal entries being non-zero. Prove that the eigenvalues of A are distinct.

4. Let A be a square matrix, which is not necessarily Hermitian. Prove that a complex number z is a Rayleigh quotient of A if and only if it is a diagonal entry of $Q^\star AQ$ for some unitary matrix Q.

5. (**Heath**)

   (a) Implement power iteration to compute the dominant eigenvalue and a corresponding eigenvector of the matrix

   $$\begin{bmatrix} 2 & 3 & 2 \\ 10 & 3 & 4 \\ 3 & 6 & 1 \end{bmatrix}.$$

   As starting vector, take $x_0 = [0\ 0\ 1]^T$.

   (b) Implement inverse iteration with a shift to compute the eigenvalue nearest to 2, and the corresponding eigenvector of the matrix

   $$\begin{bmatrix} 6 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

   Now, solve the same problem with Rayleigh Quotient Iteration : do you see any speedup in convergence ? You should use the code for QR factorization which was implemented in the previous homework.

6. (**Heath**) Write a program to find the minimum of each of the following functions on the interval [0,3]. Draw a plot of each function to confirm that it is unimodal.

   (a) $f(x) = x^4 - 14x^3 + 60x^2 - 70x$

   (b) $f(x) = 0.5x^2 - \sin(x)$

   (c) $f(x) = x^2 + 4\cos(x)$

   (d) $f(x) = \Gamma(x)$

7. (**Heath**) Write a program to find a minimum of Rosenbrock's function

$$f(x, y) = 100(y - x^2)^2 + (1 - x)^2$$

   using each of the following methods : (i) Steepest descent, (ii) Newton, (iii) Damped Newton. You should try each of the methods from each of the starting points $[-1 \;\; 1]^T$, $[0 \;\; 1]^T$, $[2 \;\; 1]^T$. For any line searches and linear system solutions required, you may use MATLAB routines. Plot the path taken in the plane by each of the methods for each of the starting points.