COL 702

Homework V

Due on Nov. 4, 2019

Justify your answers with proper reasonings/proofs.

1. Let (u, v) be a directed edge in arbitrary flow network G. Prove that if there is a minimum (s,t)-cut (S,T) such that $u \in S$ and $v \in T$, then there is no minimum cut (S', T') such that $u \in T', v \in S'$). Note that by definition of cut, $s \in S, t \in T$, and similarly $s \in S', t \in T'$.

2. Let G be an undirected graph. For a subset S of vertices, let $e(S)$ denote the number of edges which have both the end-points in S. Given a rational number α, we would like to find out if there us a subset S of vertices such that $\frac{e(S)}{|S|} \geq \alpha$. Give an efficient algorithm to solve this problem.

3. Let G be an undirected graph and s and t be two special vertices in it. Give an efficient algorithms to find the maximum number of node disjoint paths from s to t (a set of paths from s to t are said to be node-disjoint if no two of them share a vertex other than s or t).

4. As in the case of max-flow, give a capacity scaling algorithm for the cycle cancelling algorithm for min cost flow. Analyze the running time of your algorithm.

5. You are given a directed graph G where vertices have demands d_v and edges have capacities u_e. You would like to set up a flow in G such that the total outflow minus the total inflow at every vertex is equal to d_v (which could be positive or negative). Show that such a flow does not exist if and only if there is a subset S of vertices such that the total capacities of edges in $\delta^+(S)$ is strictly less than $d(S)$. Here $d(S)$ denotes $\sum_{v \in S} d_v$, and $\delta^+(S)$ denotes the set of directed edges (u, v) for which $u \in S, v \notin S$.

6. For a sequence of n days, you are given subsets $S_1, S_2, \ldots, S_n$ of $\{1, 2, \ldots, k\}$. Think of S_i as the subset of people who are available for work on day i. You need to pick exactly one person from S_i for each of the days $i = 1, \ldots, n$. For a person j, let Δ_j denote the $\sum_{i: j \in S_i} \frac{1}{|S_i|}$. This is the expected number of times j would be picked if we pick a random person from S_i on each of days $i = 1, \ldots, n$. A selection of persons, one from each set S_i, is said to be good if each person j is picked at most $\lceil \Delta_j \rceil$ times. Show that such a selection is always possible, and give an efficient algorithm to find such a selection.