Timed Bisimulations as Parameterised Bisimulations

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Abstract

We characterise Timed Bisimulations as Parameterised Bisimulation, therefore obtaining its logical characterisation over TLTS generated from Timed Automata.

1 Parameterised Modal Logic

▶ **Definition 1.1.** The syntax of the logic $\mathcal{L}_{(\rho,\sigma)}$ is given by the following BNF

$$\varphi \ := \ \top \mid \bot \mid \langle a\rangle^{\rho}\varphi \mid [a]^{\sigma^{-1}}\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2$$

where $a \in \mathcal{A}ct$. The semantics of $\varphi \in \mathcal{L}_{(\rho,\sigma)}$ defined inductively as

$$\begin{array}{lclclcl} \|\top\|^{\mathcal{P}} & = & \mathcal{P} & \quad & \|\varphi_1 \vee \varphi_2\|^{\mathcal{P}} & = & \|\varphi_1\|^{\mathcal{P}} \cup \|\varphi_2\|^{\mathcal{P}} \\ \|\bot\|^{\mathcal{P}} & = & \emptyset & & \|\varphi_1 \wedge \varphi_2\|^{\mathcal{P}} & = & \|\varphi_1\|^{\mathcal{P}} \cap \|\varphi_2\|^{\mathcal{P}} \end{array}$$

$$\begin{array}{lcl} \|\langle a\rangle^{\rho}\varphi\|^{\mathcal{P}} & = & \{p \mid \exists b, p'[a\rho b \wedge p \xrightarrow{b} p' \wedge p' \in \|\varphi\|^{\mathcal{P}}]\} \\ \|[a]^{\sigma^{-1}}\varphi\|^{\mathcal{P}} & = & \{p \mid \forall b, p'[b\sigma a \wedge p \xrightarrow{b} p' \Rightarrow p' \in \|\varphi\|^{\mathcal{P}}]\} \end{array}$$

▶ **Theorem 1.2.** *If ρ, σ are preorders then $\sqsubseteq_{(\rho,\sigma)} = \preceq_{\mathcal{L}_{(\rho,\sigma)}}$, i.e. $\mathcal{L}_{(\rho,\sigma)}$ is a logical characterisation of $\sqsubseteq_{(\rho,\sigma)}$ over (ρ,σ)-image-finite LTS.* ◀

2 Timed Automata

▶ **Definition 2.1.** An LTS $\mathcal{T} = \langle \mathcal{P}, \mathcal{A}ct \cup \mathbb{R}_{\geq 0}, \longrightarrow\rangle$ is a timed labelled transition system (TLTS) if it satisfies the following conditions

- *(Determinism)* If $s \xrightarrow{d} s'$ and $s \xrightarrow{d} s''$, for any delay $d \in \mathbb{R}_{\geq 0}$, then $s' = s''$.
- *(Additivity)* If $s \xrightarrow{d_1} s'$ and $s' \xrightarrow{d_2} s''$, then $s \xrightarrow{d_1+d_2} s''$.
- *(0-delay)* For every $s \in \mathcal{P}$, $s \xrightarrow{0} s'$ iff $s = s'$.

▶ **Definition 2.2.** A timed automata [1] is a structure $\mathcal{A} = (\mathcal{N}, \mathcal{A}ct, \mathcal{C}, \rightarrow)$, where $\mathcal{N}$ is a finite set of nodes, $\mathcal{A}ct$ is a finite set of action, and $\mathcal{C}$ is a finite collection of clocks. A transition from node n to n' is given as $n \xrightarrow{r,g,a} n'$, where r is the subset of clocks being reset, guard g is the conjunction of formulas of the form $x \bowtie v$ or $x - y \bowtie v$, with $x, y \in \mathcal{C}$ and $v \in \mathbb{N}$, and $a \in \mathcal{A}ct$ is the label.

▶ **Definition 2.3.** The behaviour of a timed automata $\mathcal{A} = (\mathcal{N}, \mathcal{A}ct, \mathcal{C}, \rightarrow)$ is described by the TLTS $\mathcal{T}_{\mathcal{A}} = \langle \mathcal{N} \times \mathbb{R}^{\mathcal{C}}_{\geq 0}, \mathcal{A}ct \cup \mathbb{R}_{\geq 0}, \longrightarrow\rangle$ where

$$\begin{array}{lll} (n, v) \xrightarrow{a} (n', v') & \text{iff} & n \xrightarrow{a,g,r} n' \text{ s.t. } v \vDash g \wedge v' = v[r \mapsto 0] \\ (n, v) \xrightarrow{d} (n, v') & \text{iff} & v' = v + d \end{array}$$

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The set of delays can be extended with a symbolic value δ, denoted $\mathbb{R}^{\delta}_{\geq 0}$, which denotes the least possible delay strictly greater than 0, but less than any other real value. More formally, for any $x \in \mathbb{R}_{>0}$, we have $x - \delta > x > x + \delta$, along with values 0 and δ as the set $\mathbb{R}^{\delta}_{\geq 0}$. The usual rules of addition are extended with δ arithmetic, which follows from the two basic rules, $\delta + \delta = \delta$ and $\delta - \delta = 0$. The δ-TLTS follows all the properties of TLTS, with delays belonging to the set $\mathbb{R}^{\delta}_{\geq 0}$. We can also give the semantics of a timed automata through the δ-TLTS, in the same way as definition 2.3.

2.1 Timed Bisimulations

Besides timed bisimulation, which is just strong bisimulation over TLTS, there are few other interesting bisimulations in the timed setting, one being time abstracted bisimulation, which allows a delay to be matched with any other delay, and timed prebisimulation, which defines a faster than relation, requiring the faster process to always match with a lesser delay. Both can be formulated as parameterised bisimulations yielding their logical characterisation, but since the application to timed prebisimulation is more challenging, besides it being a preorder, we will focus on it here.

2.1.1 Timed Prebisimulation:

Let $\gtrsim = Id_{\mathcal{A}ct} \cup \geq_{\mathbb{R}}$, i.e. for any $u, v \in \mathcal{A}ct \cup \mathbb{R}$, we have $u \gtrsim v$ iff $u = v$ if $u, v \in \mathcal{A}ct$ or $u \geq v$ if $u, v \in \mathbb{R}$. Timed prebisimulation is the $(\gtrsim, \gtrsim)$-bisimulation over TLTS.

We will restrict ourselves to one clock timed automata, as it has a very useful property which plays an important role in the decidability of timed prebisimulation [4].

▶ **Definition 2.4.** The clock valuations $v, v' \in \mathbb{R}^{\delta}_{\geq 0}$ are indistinguishable w.r.t to the timed automata $\mathcal{A}$, denoted $v \equiv_{\mathcal{A}} v'$, iff for every transition $n \xrightarrow{g,a,r} n'$ in $\mathcal{A}$, we have $v \vDash g$ iff $v' \vDash g$.

▶ **Lemma 2.5.** *For any location n in a one clock timed automata $\mathcal{A} = (\mathcal{N}, \mathcal{A}ct, \{x\}, \rightarrow)$, if $v \equiv_{\mathcal{A}} v + d$ for some $v, d \in \mathbb{R}^{\delta}_{\geq 0}$, then $\langle n, v\rangle \sqsubseteq_{(\gtrsim,\gtrsim)} \langle n, v + d\rangle$.*

Proof. Let relation $\mathcal{R}$ be $\{(\langle n, v\rangle, \langle n, v + d\rangle) \mid v \equiv_{\mathcal{A}} v + d, v, d \in \mathbb{R}^{\delta}_{\geq 0}, n \in \mathcal{N}\}$. We will show that $\mathcal{R}$ is a timed prebisimulation. Let us denote $p = \langle n, v\rangle$ and $q = \langle n, v + d\rangle$ such that $p\mathcal{R}q$ holds. Then we have the following two cases

- *delay transitions:* For any d' such that $q \xrightarrow{d'} q'$, we have $p \xrightarrow{d+d'} q'$, where $q'\mathcal{R}q'$ always holds. Now if $p \xrightarrow{d'} p'$, we have two cases-$d' < d$, in which case $p' \;\mathcal{R}\; q$ holds and q can make 0-delay transition to itself or $d' \geq d$, then $q \xrightarrow{d'-d} p'$ holds and $p' \;\mathcal{R}\; p'$ always holds.
- *label transitions:* Since $v \equiv_{\mathcal{A}} v + d$, any action transition that is enabled for p will also be enabled for q, so for any $p \xrightarrow{a} p'$ we will also have $q \xrightarrow{a} q'$, where p' and q' have the same location. Now if it is a reset action, then $p' = q'$, otherwise the clock values will remain same. In both cases $p' \;\mathcal{R}\; q'$ holds.

◀

As a consequence of this lemma, an automaton with n guarded transition will have its clock values divided into n+1 intervals, with the values from the same interval being indistinguishable. The state whose clock value, say v_i, is the starting point of the interval will be prebisimilar to all states with same location and having clock value, say v', in that interval, i.e. $(l, v_i)\sqsubseteq_{(\gtrsim,\gtrsim)}(l, v')$. Similarly, we will have $(l, v')\sqsubseteq_{(\gtrsim,\gtrsim)}(l, v_f)$ where v_f is ending value of the interval. But if the interval is open, then v_i or v_f may not exist, which is why we need δ-TLTS, where these values are always guaranteed to exist. Therefore, the δ-TLTS for one clock timed automata can be abstracted to a finite LTS, by only retaining clock values which are starting or ending point of some interval. The abstracted LTS will be polynomial in the size of the timed automata for one clock. The relation $\gtrsim$ can be extended to $\mathbb{R}_\delta$, as $\gtrsim^\delta$, to define timed prebisimilarity, $\sqsubseteq_{(\gtrsim^\delta,\gtrsim^\delta)}$, over δ-TLTS, where $\mathcal{L}_{(\gtrsim^\delta,\gtrsim^\delta)}$ will be its logical characterisation by theorem 1.2. Infact we can show that prebisimilarity on TLTS is preserved in δ-TLTS, thereby obtaining $\mathcal{L}_{(\gtrsim^\delta,\gtrsim^\delta)}$ as the logical characterisation of timed prebisimulation over timed automata.

▶ **Theorem 2.6.** *Let $\mathcal{A}$ be any timed automata. Then for any states $p, q \in \mathcal{T}_\mathcal{A}$, we have $p \sqsubseteq_{(\gtrsim,\gtrsim)} q$ in $\mathcal{T}_\mathcal{A}$ iff $p \sqsubseteq_{(\gtrsim^\delta,\gtrsim^\delta)} q$ in $\mathcal{T}^\delta_\mathcal{A}$.*

Proof. It can be easily seen that $\mathcal{T}_\mathcal{A}$ can be embedded in $\mathcal{T}^\delta_\mathcal{A}$, such that for any state p with real clock value, only a δ delay will take it to a state with $\mathbf{R}^\delta$ clock value, and vice-versa. Using this fact, one can show that if $p \sqsubseteq_{(\gtrsim^\delta,\gtrsim^\delta)} q$ holds in $\mathcal{T}^\delta_\mathcal{A}$ then $p \sqsubseteq_{(\gtrsim,\gtrsim)} q$ in $\mathcal{T}_\mathcal{A}$. For the converse, we need our logical characterisation results, which show that $\mathcal{L}_{(\gtrsim^\delta,\gtrsim^\delta)}$ will characterise $\sqsubseteq_{(\gtrsim^\delta,\gtrsim^\delta)}$ by theorem 1.2. Now any formula $\langle d+\delta\rangle^{\geq}\psi$ can also be written as $\bigvee_{d'\in\mathbf{R}|d'>d}\langle d'\rangle^{\geq}\psi$. Though this requires an infinite disjunction. Similar reduction can also be made for box modality using infinite conjunction. Hence, every formula in $\mathcal{L}_{(\gtrsim^\delta,\gtrsim^\delta)}$ can be reduced to a formula in $\mathcal{L}^\infty_{(\gtrsim,\gtrsim)}$. By logical characterisation results, this implies that for any p, q such that $p \sqsubseteq_{(\gtrsim,\gtrsim)} q$, i.e. every formula of $\mathcal{L}^\infty_{(\gtrsim,\gtrsim)}$ which is satisfied by p will also be satisfied by q, then they will also preserve this inclusion for $\mathcal{L}_{(\gtrsim^\delta,\gtrsim^\delta)}$ formulas, and hence $p \sqsubseteq_{(\gtrsim^\delta,\gtrsim^\delta)} q$. ◀

The logic $\mathcal{L}_{(\gtrsim^\delta,\gtrsim^\delta)}$ can be interpreted over TLTS as well, as the extension of $\mathcal{L}_{(\gtrsim,\gtrsim)}$, with the modal operators $\langle d-\delta\rangle^{\geq} = \langle d\rangle^{\geq}$, $\langle d+\delta\rangle^{\geq} = \langle d\rangle^{>}$, $[d-\delta]^{\leq} = [d]^{<}$, and $[d+\delta]^{\leq} = [d]^{\leq}$. Equivalently the following is a logical characterisation for timed prebisimulation

$$\varphi \ := \ \langle a\rangle\varphi \mid [a]\varphi \mid \langle d\rangle^{\geq}\varphi \mid [d]^{\leq}\varphi \mid \langle d\rangle^{>}\varphi \mid [d]^{<}\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2$$

Since the abstracted LTS for timed prebisimulation is finite, model checking for this logic is decidable, and we obtain the algorithms for generating not only the distinguishing formulae, but also the characteristic formula by extending this logic with fixed point operators.

References

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