COL 752, Geometric Algorithms: Problem Set 1

1. Design an $O(n \log n)$ algorithm for the three dimensional maximal problem for an input set $S = (x_i, y_i, z_i)\ i \leq n$.
   Hint: Use a generalization of line sweep and maintain the information about the 2D maximal layer.

2. **Implement** an $O(n \log n)$ algorithm for computing the area of the union of n isothetic (axis parallel) rectangles.
   Note: The choice of the language is yours.

3. Given a set of n numbers, design a $O(n \log n)$ algorithm for computing the number of *inversions*. Two numbers x_i, x_j are *inverted* if $i < j$ and $x_i > x_j$.

4. For a given set S of n points let $CH(S)$ be the convex hull of S. Consider the line-segment $s_{i,i+1}$ connecting the consecutive boundary points i and $i+1$ and the corresponding half-plane $H_{i,i+1}$ defined the by line $l_{i,i+1}$ containing the remaining points. (A boundary edge is such that the remaining points lie on one side of the line containing this edge).

   Now consider the intersection of all such half-planes $R = \cap_i H_{i,i+1}$. Prove that $R = CH(S)$.
   Hint: Half-planes are convex and intersection of convex sets are convex and show that $CH(S) \subseteq R$. Argue that every point in C can be generated by convex linear combination of two points in the CH(S) and therefore $R \subseteq CH(S)$

5. **Caratheodory theorem** Show that any point p in the convex hull of $n > d$ points in d dimensions can be expressed as a clc of a subset of $d + 1$ points.
   Hint: If $Q = \{q_1, q_2 \ldots q_{d+2}\}$ is a set of $d + 2$ points in d dimensions then there exists constants $a_1, a_2 \ldots a_{d+2}$, not all of them 0 such that $\sum_i q_i \cdot a_i = 0$ and $\sum_i a_i = 0$. Now apply this result to decrease the boundary points that contain p.

6. If you run Graham's scan on a *non-monotone* chain, show that it may not succeed by a counterexample that produces a self intersecting polygon.

7. Describe an $O(n)$ time algorithm to compute the intersection of convex polygons C_1 and C_2 where n is the cumulative number of vertices.

8. Given two convex polygons P_1 and P_2 with n_1 and n_2 vertices respectively.
   (i) Design a $O(n)$ algorithm to compute the convex hull of $P_1 \cup P_2$ where $n = n_1 + n_2$.
   (ii) Design a $O(\log n)$ time algorithm for the above problem when P_1 and P_2 are separated by a vertical line. Describe the appropriate data structures to accomplish this.

9. Design a binary-search based strategy to find the closest vertex of a convex polygon from a given input line ℓ.
   Hint: The distances of the vertices from the line increase and subsequently decrease. Also note that if ℓ intersects the polygon, then one of the two edges that ℓ intersects contain the closest vertex.

10. Prove an $\Omega(n \log h)$ lower bound for the 2D maxima problem where n and h are the input and output sizes respectively.
    Hint: Use a construction similar to the convex hull lower bound.

11. Complete the algorithmic details, including data structures needed to implement the *insertion hull* algorithm in $O(n \log n)$ steps.

12. Given a set of points P on the plane, design an efficient algorithm to compute the *diameter* $D(P)$ and width $W(P)$. The diameter of a P is defined as $\max_{x,y \in P} dist(x, y)$.
The width is defined as the minimum distance between two parallel lines such that all points are lie in the region between the parallel lines.
Hint: Prove that the pairs of points that determine the diameter and the width respectively are points on the convex hull. Modify the line sweep to an angular sweep method (also called the rotating callipers method).