Randomized Techniques in Computational Geometry

<u>I Fundamentals</u>

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Outline

- Kind of Problems
- Kind of Algorithms
- Random Sampling in Geometry
- Incremental Construction

Primary Source (for this talk)

- **Clarkson and Shor**, *"Applications of random sampling in computational geometry"*. DCG 89.
- **Reif and Sen** *"Optimal parallel randomized algorithms for 3-D hulls and related problems.* SIAMJC 92.
- **Mulmuley and Sen**, *Dynamic point location in arrangements of hyperplanes.* DCG 92.
- **Mulmuley** *"Computational Geometry : An introduction through randomized algorithms"* . Prentice Hall 94.

The problems addressed

- Range searching
- Ray shooting/Point location
- Planar partitioning
- Convex hulls
- Linear programming
- Nearest neighbour

The problems addressed cont'd

- Triangulation
- Hidden surface
- Levels of arrangements
- Diameter/ Width
- Euclidean minimum spanning tree

$$\frac{\text{Exact computations}}{\text{“Real” RAM}}$$

Randomization in Computational Geometry

- **Improved Complexities**

  Range searching

- **Simpler Algorithms**

  Convex hulls, triangulation, LP

- **Dynamic Algorithms**

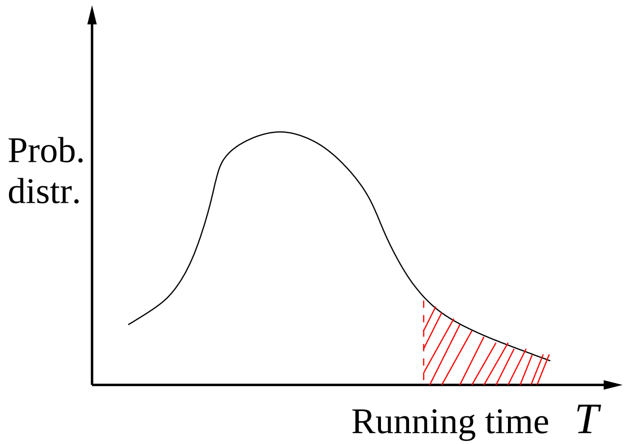
  with minimal modifications

- **Parallel Algorithms**

  Faster, more efficient

Randomized Algorithms

- RNG $O(\log n)$ bits in constant time
- Assumes **No** input distribution
- Halts with a correct output
- Running time is bounded by some probability distribution



Expectation $[T]$

Tail estimates : Prob. $[T_n > f(n)] \leq \epsilon]$

Elementary Tools

Probabilistic Inequalities :

- Markov — *only expectation*
- Chernoff — *moment generating function*
- Chebychev — *intermediate(bounding random bits)*

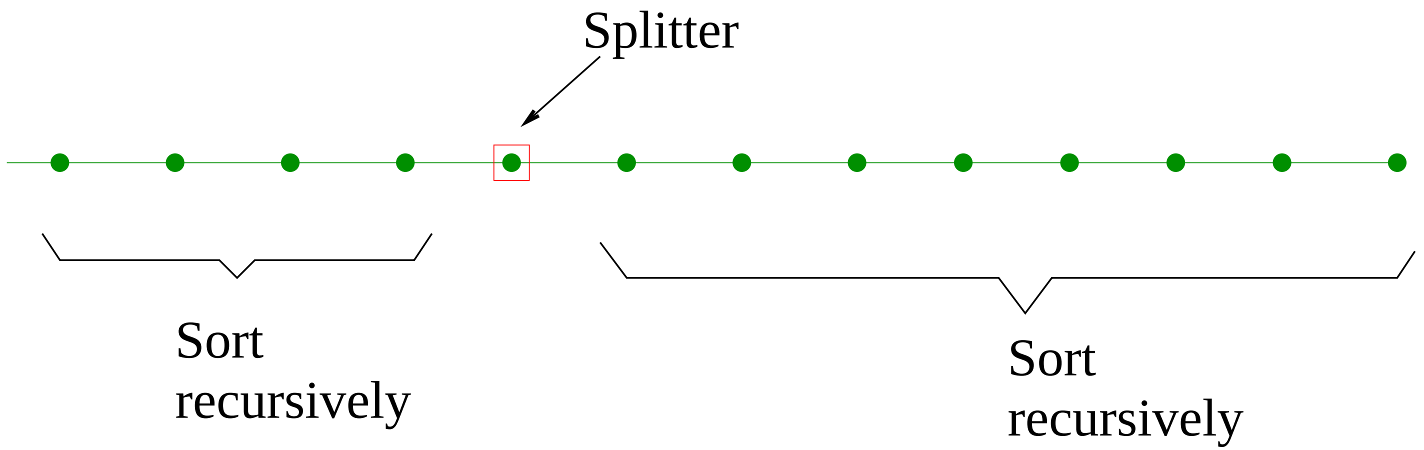
Linearity of Expectation $E[X+Y] = E[X] + E[Y]$

for any X, Y not necessarily independent

$$P(A \cup B) \leq P(A) + P(B)$$

Notation : $\widetilde{O}\,(\cdot) \stackrel{def}{=} O(\cdot)$ with prob. $1 - \frac{1}{n}$

Quick Sort



Ideal :

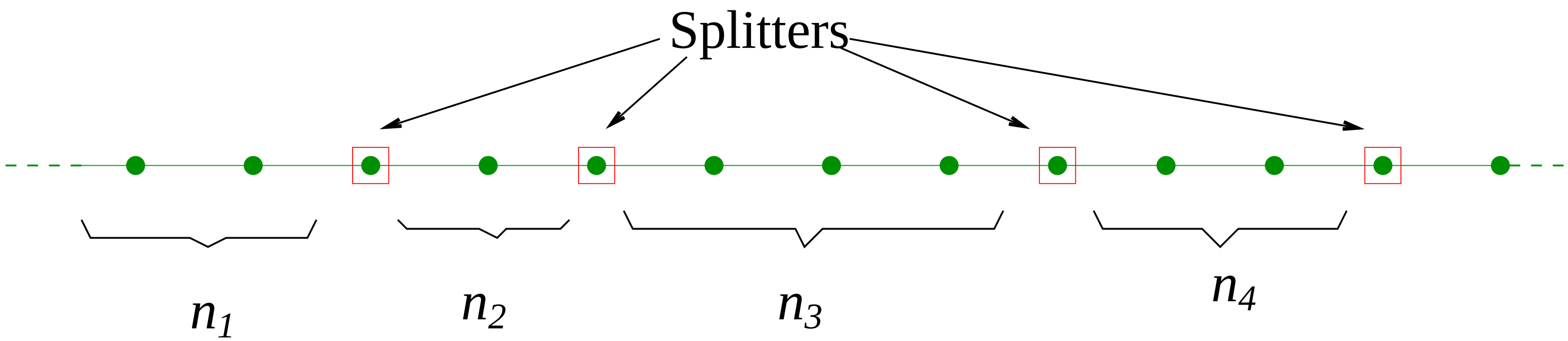
$$T(n) = 2T(n/2) + O(n) \le O(n \log n)$$

randomized : $T(n) = T(n_1) + T(n - n_1 - 1) + O(n)$

where n_1 is a random variable in $[1, n]$

$$E[T(n)] : O(n \log n)$$

Generalized : r splitters



Ideal :

$$\begin{aligned} T(n) &= \sum T(n/r) + O(n \log r) \\ &\le O(n \log n) \end{aligned}$$

$$E[T(n)] : O(n \log n)$$

Kind of sampling

Choose a random subset $R \subset N$

- with replacement
- without replacement
- Bernoulli sample
  (expected sample size is $|R|$ by picking every element with prob. $(\frac{|R|}{|N|})$).

"parallel" sampling

Remark : Little difference in final results. We shall choose the one that simplifies proof.

Some Notations

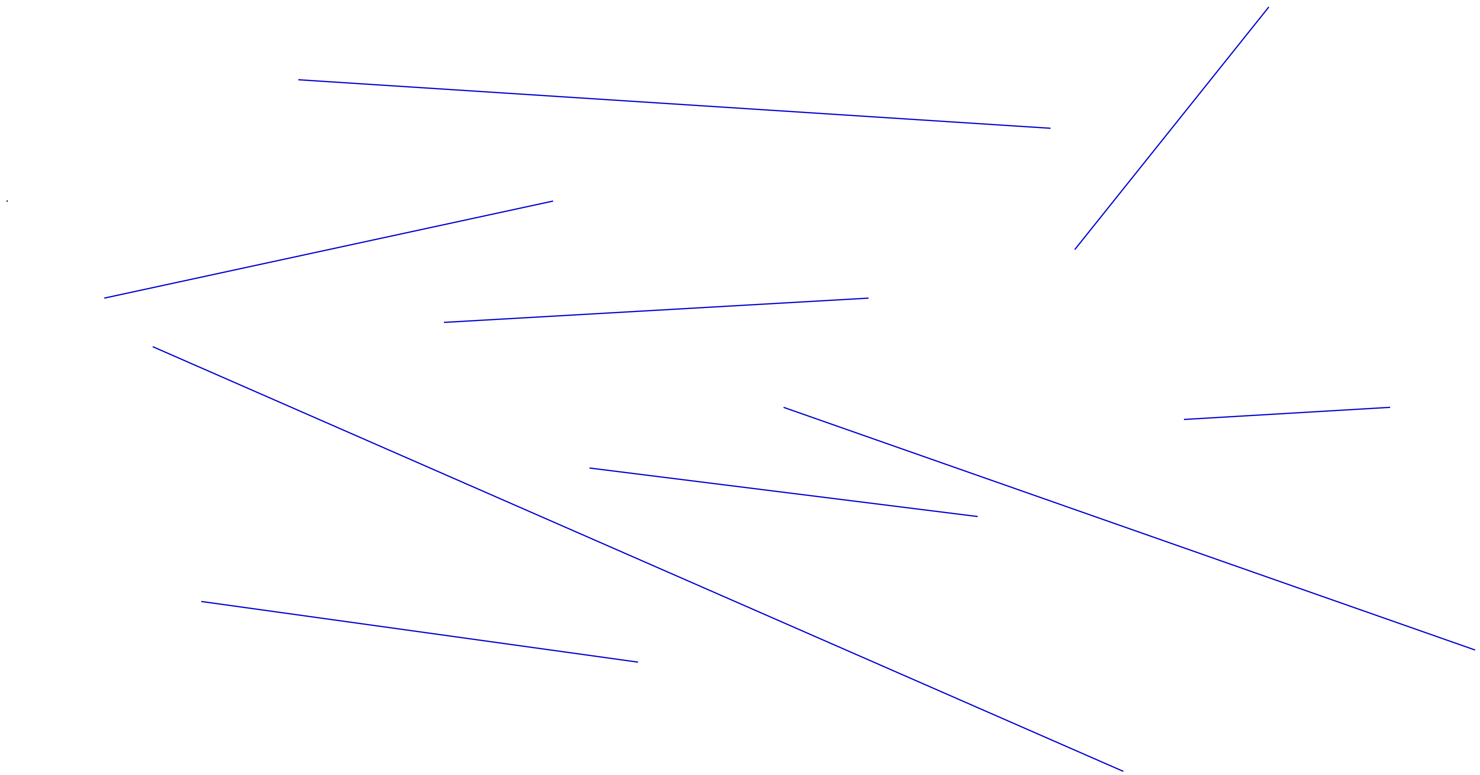
N : set of objects $|N| = n$

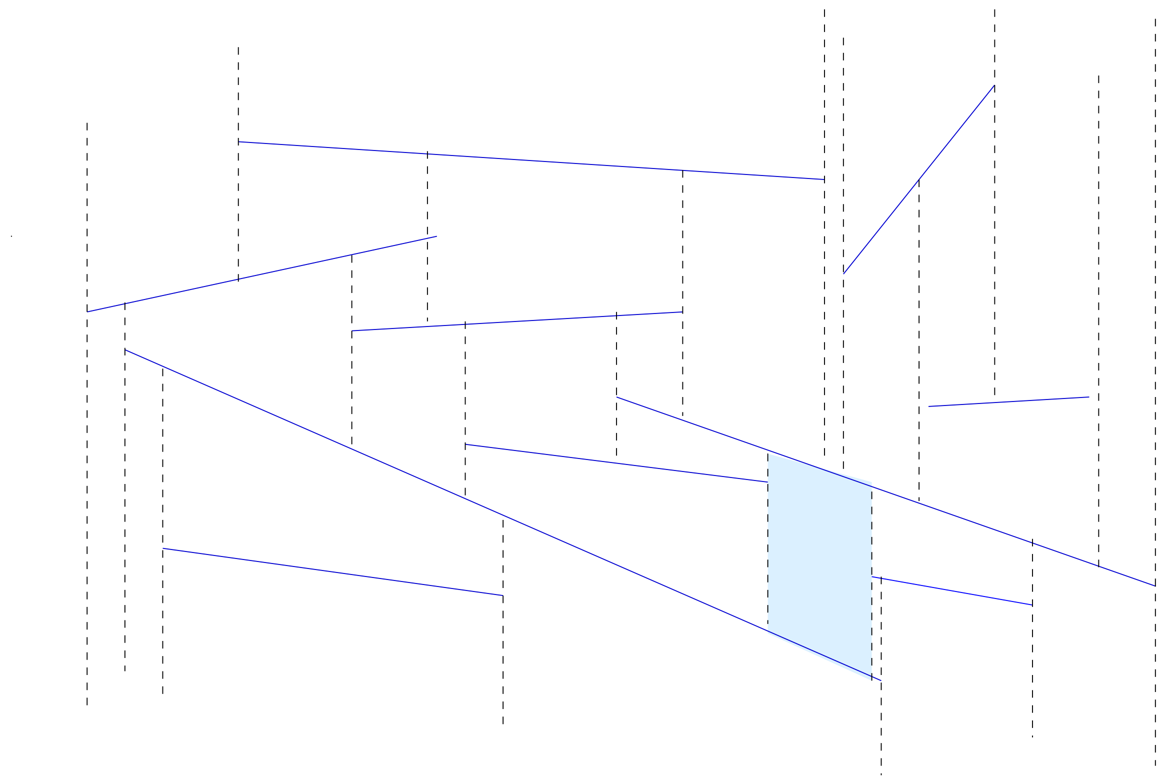
σ : $(D(\sigma)$ $L(\sigma))$ (D, L)

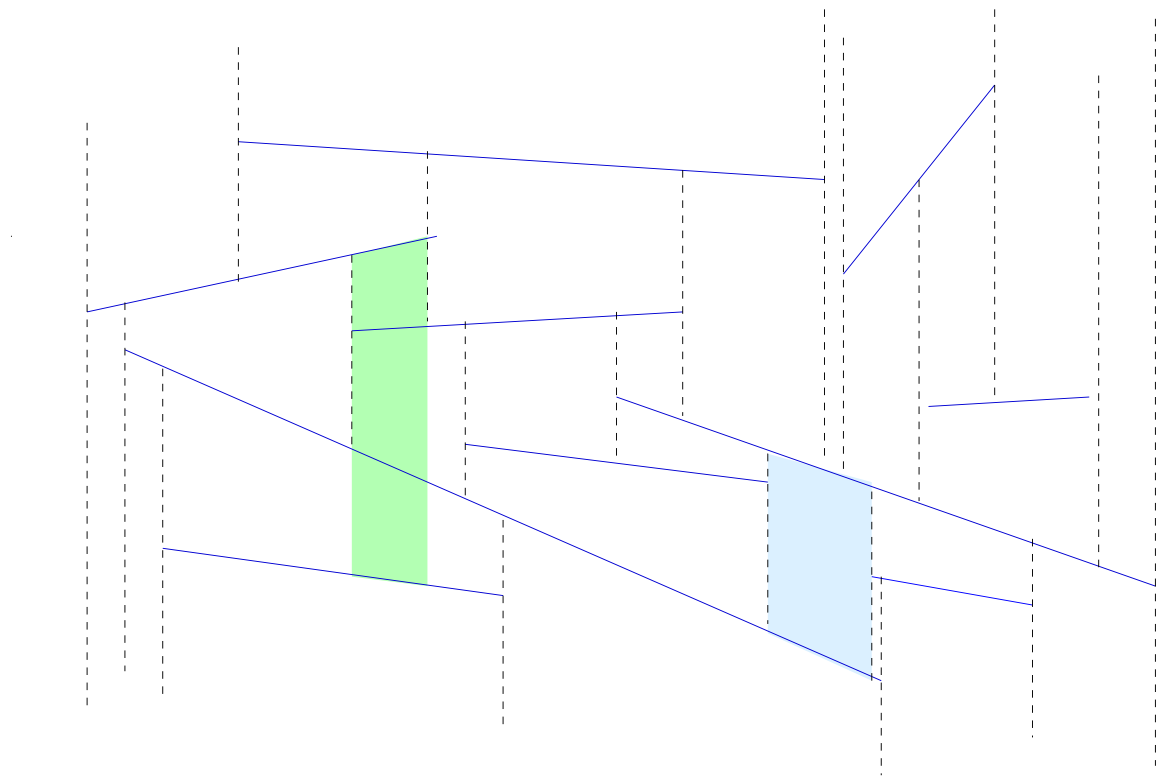
configuration define conflict

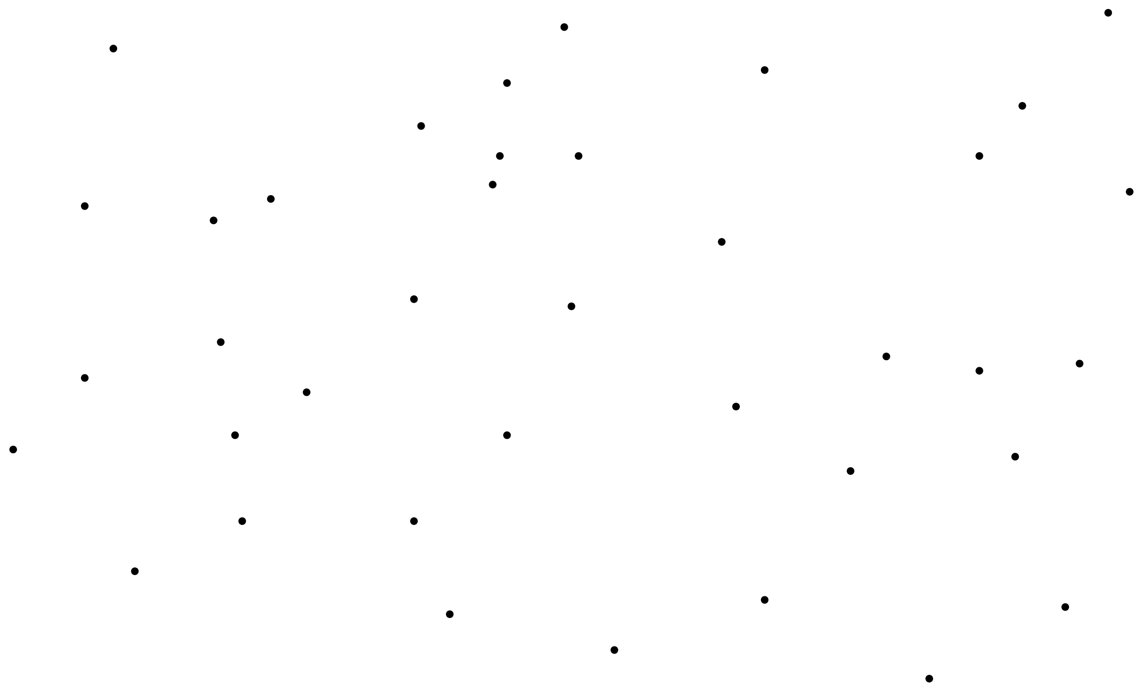
Assumptions(for technical reasons)

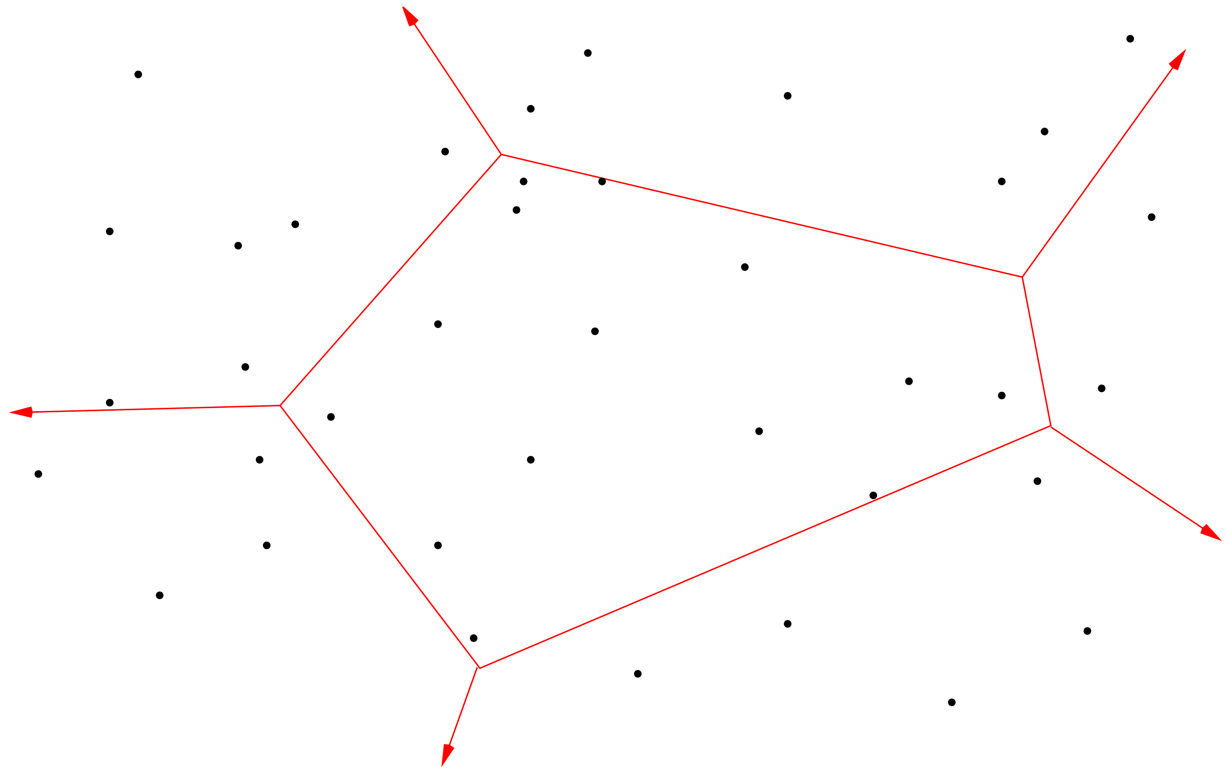
- D is bounded by constant
- *Valence* (no. of σ with same $D(\sigma)$ is bounded

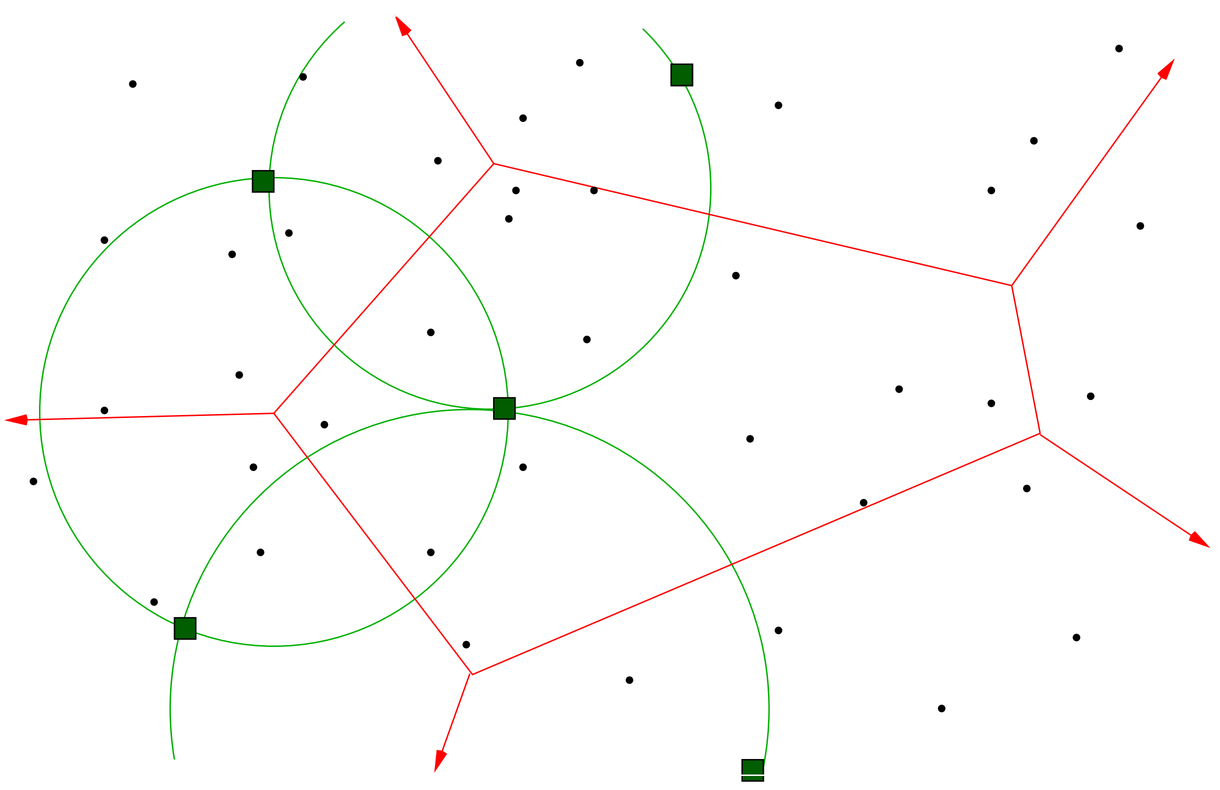


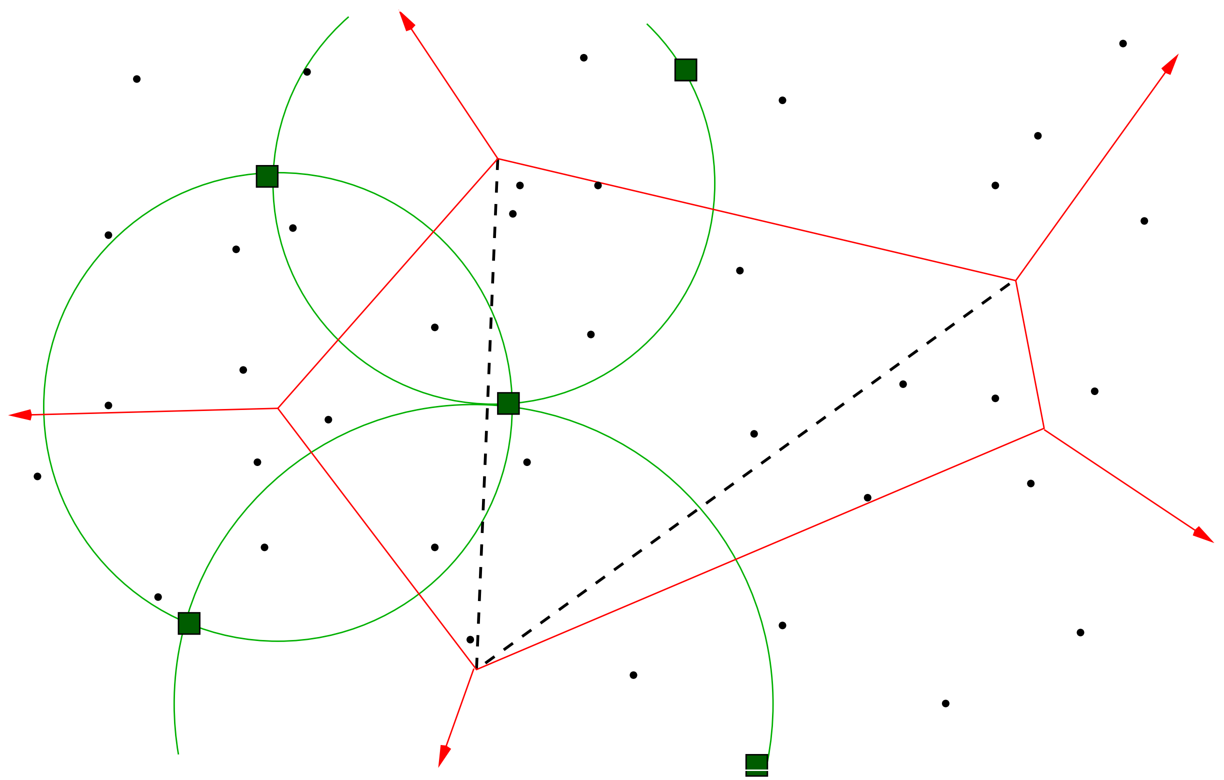






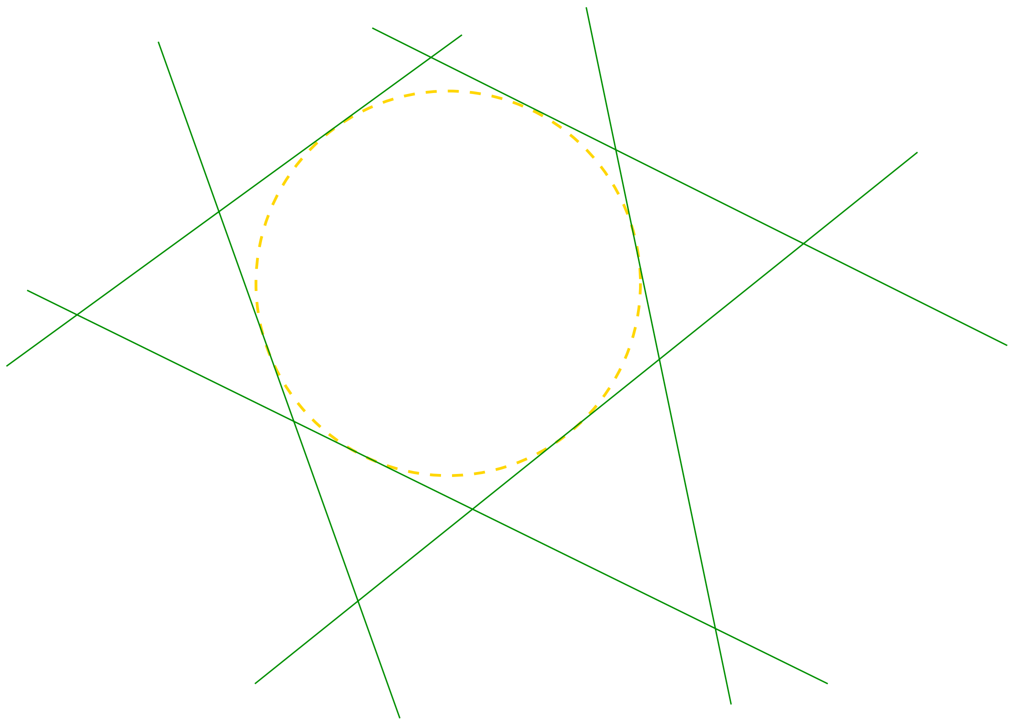






Need for bounded degree

All lines tangential to a circle



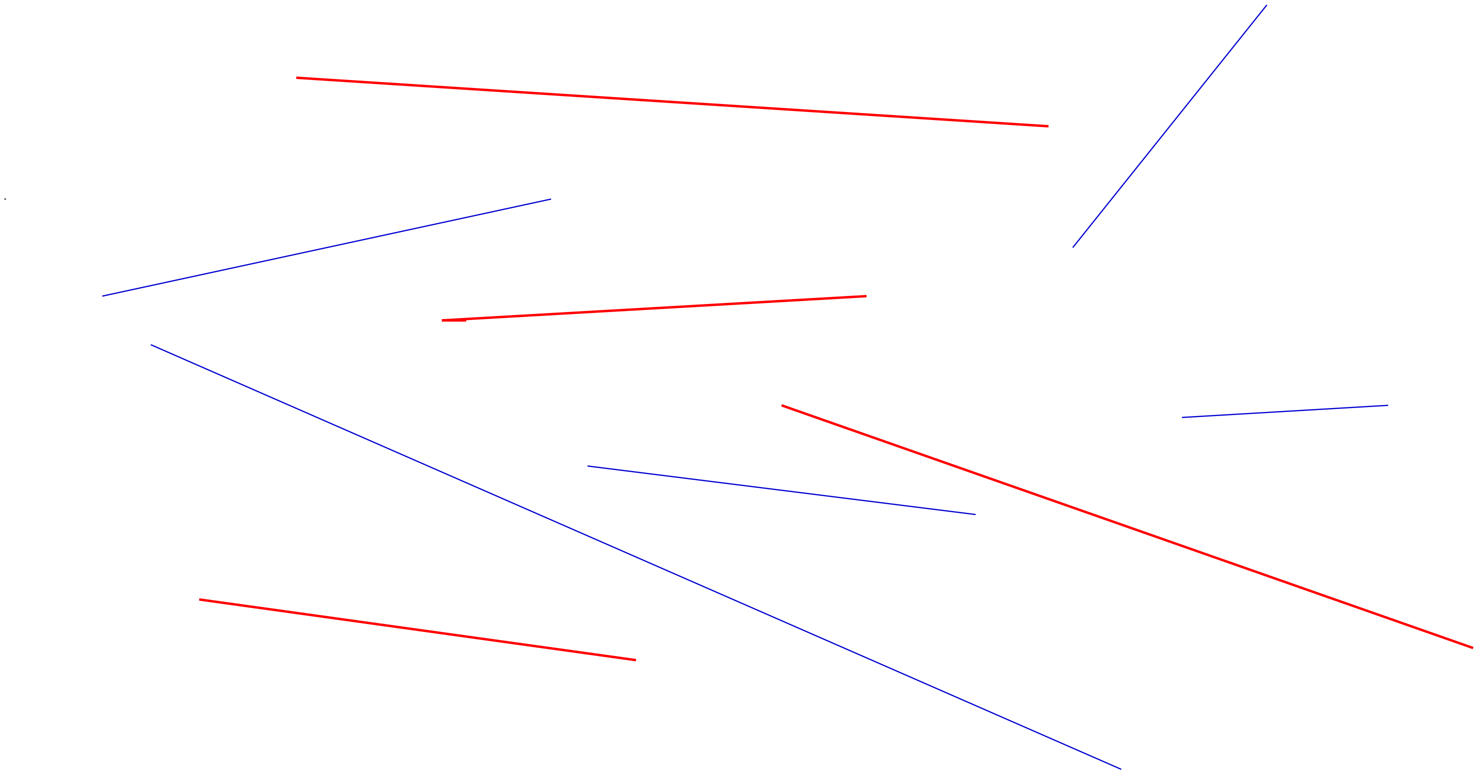
Any subset of size r induces a face that intersects $n - r$ lines.

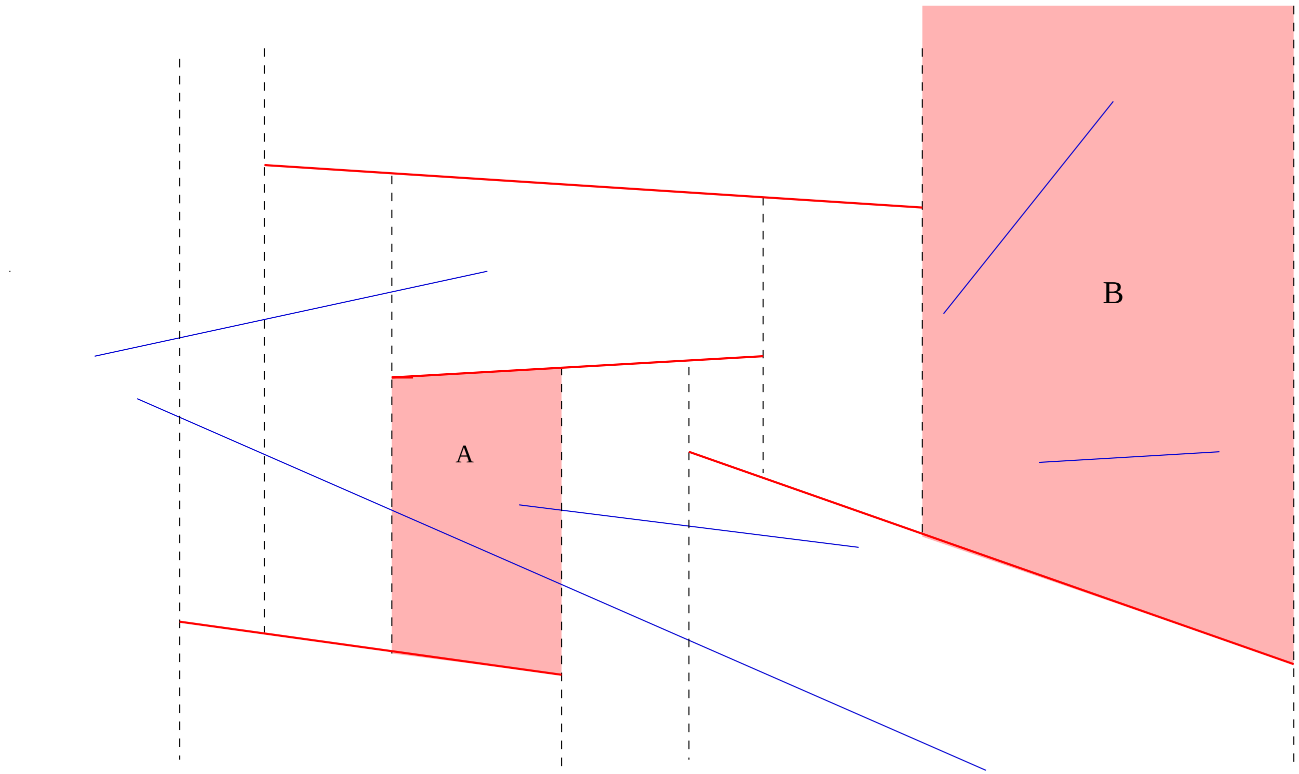
<u>Some Notations cont'd</u>

- $\Pi(N)$: set of configurations (multiset) over N
- $\Pi^i(N)$: set of configurations with conflict size $=i$
- $\Pi^0(R)$: configurations <u>active</u>

For $R \subset N$, σ is feasible (for R) if $D(\sigma) \subset R$

We shall often use $\Pi(N)$ to also denote $|\Pi(N)|$




A
B

A simple combinatorial bound

Claim $\Pi^a(N) = O(2^{a+d} \cdot E[\Pi^0(R)])$
where R is a random sample of size $n/2$.

$\Pi^0(R) = \sum_{\sigma \in \Pi(N)} I_{\sigma,R}$
where $I_{\sigma,R}$ is 1 if σ is feasible.

$E[\Pi^0(R)] = E[\sum_{\sigma \in \Pi(N)} I_{\sigma,R}] = \sum_{\sigma \in \Pi(N)} E[I_{\sigma,R}]$
$= \sum_{\sigma \in \Pi(N)} \Pr\{\sigma \in \Pi^0(R)\}$

$\geq \sum_{\sigma \in \Pi^a(N)} \{\sigma \in \Pi^0(R)\}$
$= \Pi^a(N) \cdot \frac{1}{2^{a+d}}$

Claim :

$$\Pr\{\max_{\sigma \in \Pi^0(R)} l(\sigma) \geq c\frac{n}{r}\log r\} \leq \frac{1}{2}$$

$|R| = r$ by Bernoulli sampling

$p(\sigma, r)$: conditional probability that none of the k conflicting element are selected given σ is feasible

$\leq (1 - r/n)^k$

$\leq e^{-c \log r}$ for $\boxed{k \geq cn/r \ln r}$

BAD σ

$= 1/r^c$

$q(\sigma, r)$: Prob. that $D(\sigma) \subset R$

Prob. that $\sigma \in \Pi^0(R) = p(\sigma, r) \times q(\sigma, r)$

Prob. that some $\sigma \in \Pi^0(R)$ is **<u>BAD</u>** $(l(\sigma) \geq c(n \ln r)/r)$:

$$\leq \frac{1}{r^c} \sum_{\sigma \in \Pi(N)} q(\sigma, r)$$

$$= \frac{1}{r^c} \quad E[\Pi(R)]$$

(usually $\Pi(R) = r^{O(1)}$)

$\leq 1/2$ for appropriate c

Sum of subproblem sizes

Def: c-order conflict $\left(\begin{array}{c} l(\sigma) \\ c \end{array}\right)$, for some $c \geq 0$

Let $T_c = \sum_{\sigma \in \Pi^0(R)} \left(\begin{array}{c} l(\sigma) \\ c \end{array}\right)$

Remark For technical reasons it is not $l(\sigma)^c$. $T_0 = |\Pi^0(R)|$. $T_1 =$ sum of subproblems.

Claim $E[T_c] = O(\left(\frac{n}{r}\right)^c E[\Pi^c(R)])$

For constant c, $E[\Pi^c(R) = O(E[\Pi^0(R)]$ implying that average conflict size is very close to $\frac{n}{r}$

$T_c = \sum_{\sigma \in \Pi(N)(R)} \begin{pmatrix} l(\sigma) \\ c \end{pmatrix} I_{\sigma,R}$ where $I_{\sigma,R} = 1$ if $\sigma \in \Pi^0(R)$.

$E[T_c] = \sum_{\sigma \in \Pi(N)} \begin{pmatrix} l(\sigma) \\ c \end{pmatrix} p^{d(\sigma)} \cdot (1-p)^{l(\sigma)}$ for $l(\sigma) \geq c$.

$= \sum_{\sigma \in \Pi(N)} \begin{pmatrix} l(\sigma) \\ c \end{pmatrix} p^{d(\sigma)+c} \cdot (1-p)^{l(\sigma)-c} \cdot \left(\frac{1-p}{p}\right)^c$

$\leq \left(\frac{1-p}{p}\right)^c \cdot E[\Pi^c(R)]$

since

$\Pr\{\sigma \in \Pi^c(R)\} = \Pr\{d(\sigma)$defining elements chosen and c out of
$l(\sigma)$ conflicting elements not chosen$\}$

$\leq \left(\frac{1}{p}\right)^c \cdot E[\Pi^c(R)]$

where $p = \frac{r}{n}$.

Improvements - Tail estimates ?

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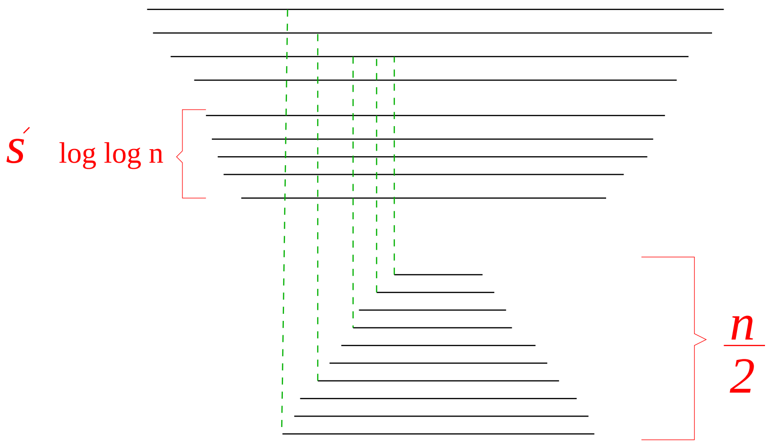


$$\Pr \ \{ \text{ no segments in S' is selected } \} \ \geq \ \left(\frac{1}{2}\right)^{\log\log n}$$

$$= \ \Omega\left(\frac{1}{\log n}\right)$$

$\Rightarrow$ With Prob. $\Omega(\frac{1}{\log n})$, # of intersections is $\Omega(n \log\log n)$

Selecting a GOOD sample w.h.p.

Motivation: In divide-and-conquer algorithms, we are often interested in bounding the maximum size of subproblems for which we need tail estimates including the sum of subproblem sizes.

Since sample is *GOOD* in the expected sense, the probability that the sum of subproblems is $\leq 2 \times E[$ sum of subproblems $]$ is $\geq 1/2$ from Markov's inequality.

Implying

If we choose a set of $\log n$ independent samples - $R_1, R_2 \ldots R_{\log n}$, at least one is **GOOD** w.h.p.

How do we know which is good ?

Polling: an efficient resampling technique

1. Choose $c \log n$ samples
2. *Poll* (sample) $S' = \frac{n}{\log^2 n}$ of the input
3. Estimate the *goodness* of the samples w.r.t. S'. Choose an R_i that is *good* w.r.t. S' (break ties arbitrarily).

Polling Lemma With high probability we obtain a *good* sample by the above procedure.

Consequences of Random Sampling

- Dynamization
- Existence of good splitters : The probabilistic method. There exists efficient derandomization by method of connditional probability and divide-and-conquer.
- Improved bounds for important combinatorial measures like $k - sets$.

Randomized Incremental Construction (RIC)

Starting from an empty set

Repeat:

1. *Insert the next object*
2. *Update the partial construction (data-structures)*

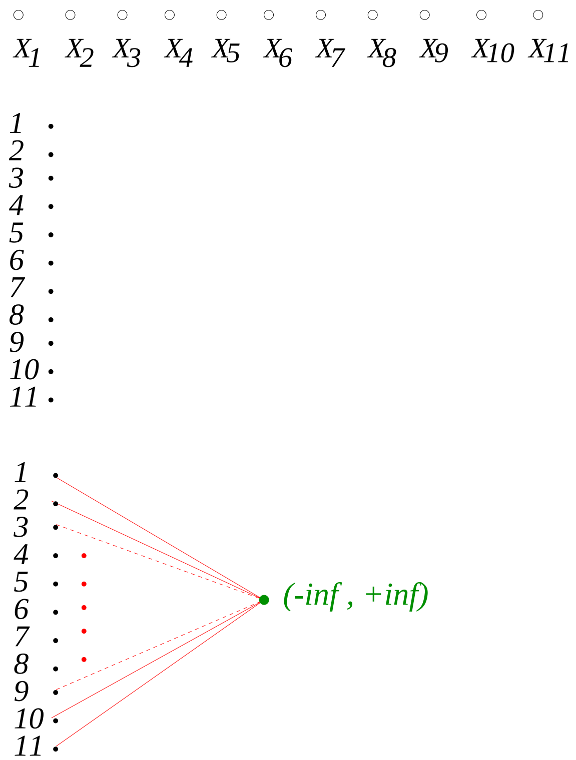
Total Time $= \sum_i$ Time to insert the i-th object.

$T_s(N) =$ Total time to insert a sequence s. (s is **good** if total time is less).

Expected total time = Expected time for a **Random Insertion sequence** (worst case for any input of size n).

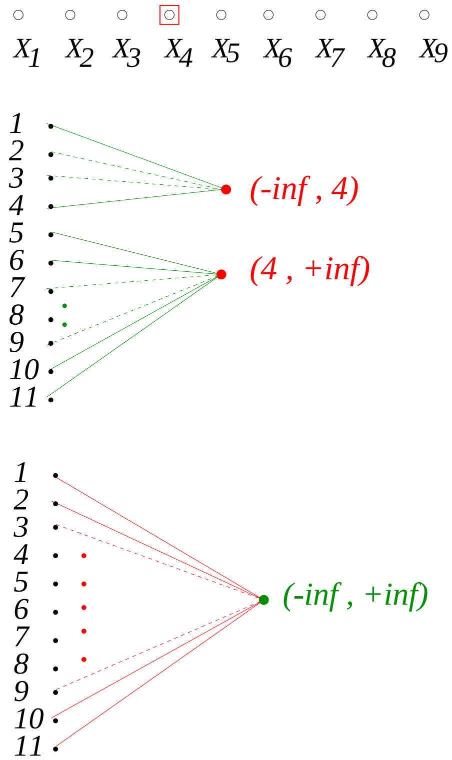
Quicksort as R.I.C.

Gradual refinement of partition



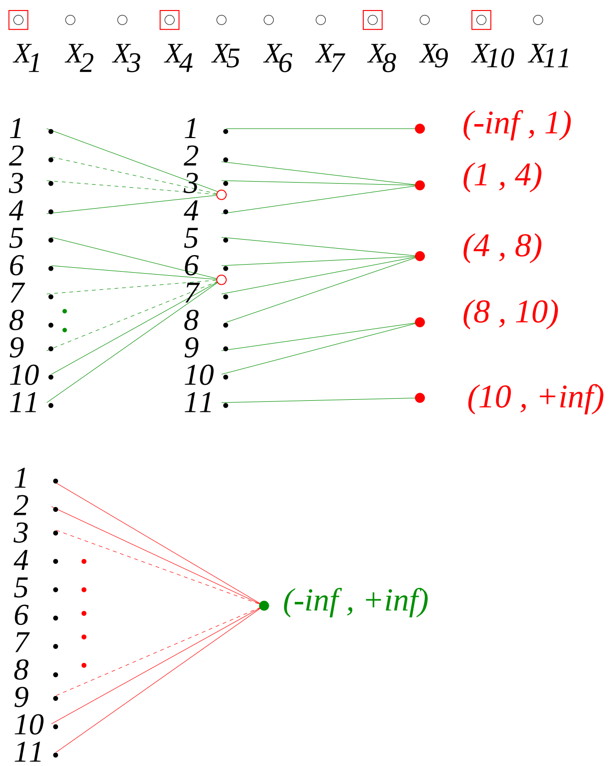
Quicksort as R.I.C.

Conflict graph



Quicksort as R.I.C.

Conflict graph



A general bound for RIC

Total (amortised) cost $= O(\text{edges created in conflict graph})$

Edges can be deleted at most once

General Step: $R \leftarrow R \cup s$ (for a fixed $R' = R \cup s$)

Expected work (#edges created **given** R')=

$$\sum_{\sigma \in \Pi^0(R \cup s)} l(\sigma) \cdot \Pr\{\sigma \in \Pi^0(R \cup s) - \Pi^0(R)\}$$

From *backward analysis* this probability equals $\frac{d(\sigma)}{r+1}$. Substituting

$$\sum_{\sigma \in \Pi^0(R \cup s)} l(\sigma) \cdot \frac{d(\sigma)}{r+1} = \frac{d(\sigma)}{r+1} \sum_{\sigma \in \Pi^0(R \cup s)} l(\sigma)$$

This is expected cost conditioned on $R' = R \cup s$.

Unconditioned Cost

$$\mathbb{E}[\ \#\ \text{edges created}\ ] = \mathbb{E}[\mathbb{E}[\ \#\ \text{edges created}\ |R']]$$

$$= \frac{d(\sigma)}{r+1}\Pr(R'\ \text{is chosen}\ )\cdot \sum_{R'} \sum_{\sigma \in \Pi^0(R')} l(\sigma) = O(\frac{d(\sigma)}{r} \cdot \frac{n}{r} E[\Pi^0(R \cup s)])$$

A common scenario $E[\Pi^0(R) = O(r)$.

$$\text{Total expected cost of RIC } = \sum_{r=1}^{r=n} O\left(\frac{d}{r} \cdot n\right)$$

$= O(n \log n)$ (applicable to convex hulls)

OPEN PROBLEM

Tail estimates

Backward Analysis

For a fixed set of $i+1$ object , what is the probability that the $i+1$-st insertion affects σ ?

Because of random insertion sequence, any one of the fixed set of $i+1$ objects is equally likely to be the last inserted object (by symmetry)

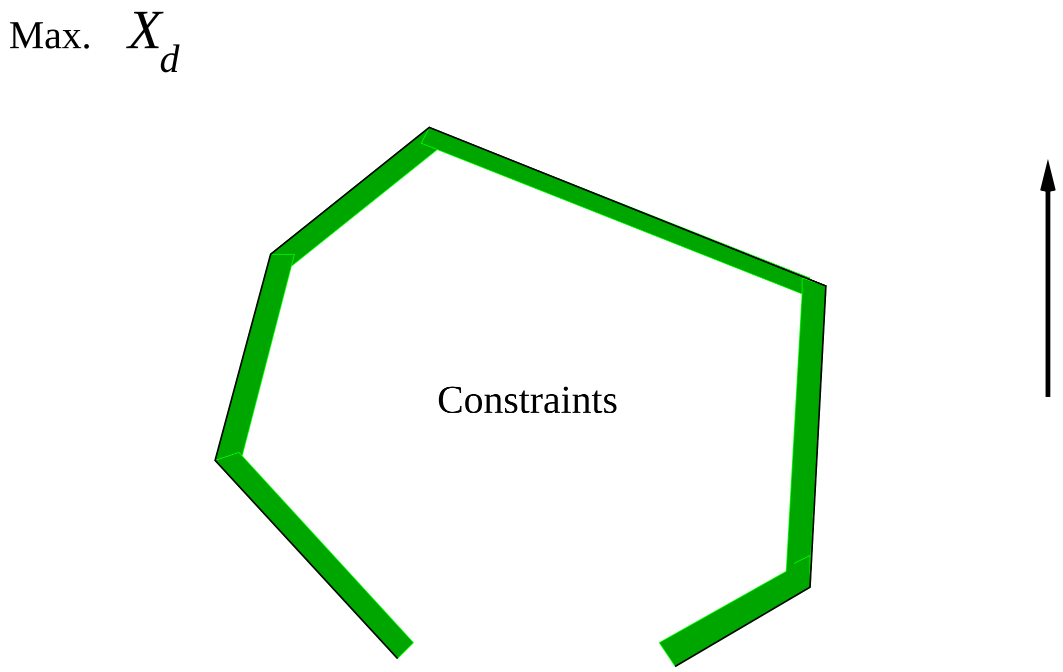
What is the probability that a random deletion from $i+1$ objects defines σ ? (pretending to run backwards).

$$\frac{d(\sigma)}{i+1}$$

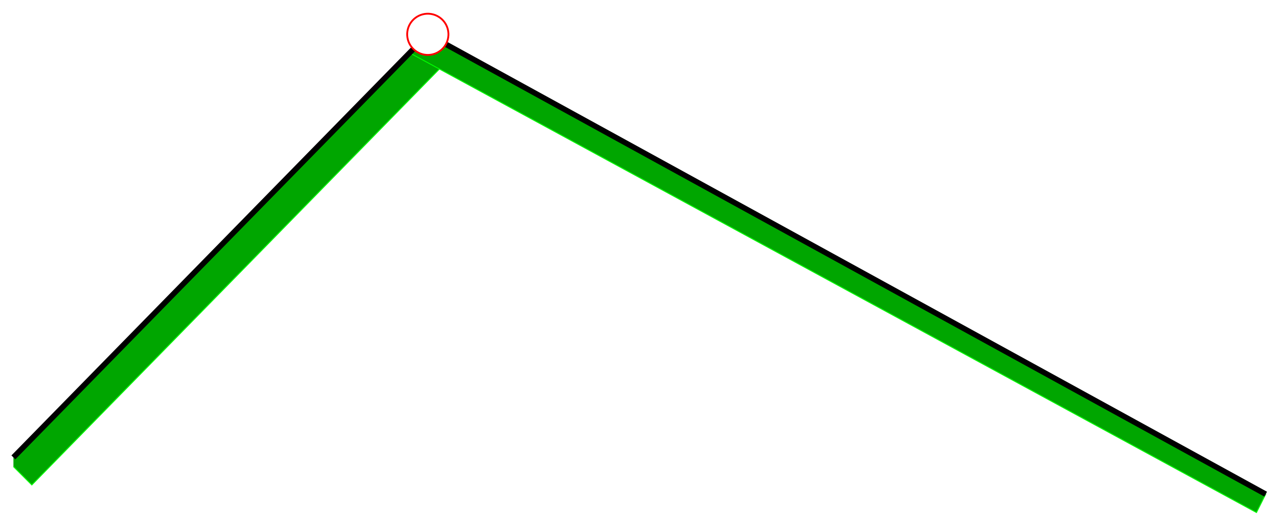
Since conditional expected cost depends only on i (independent of the actual set of objects)

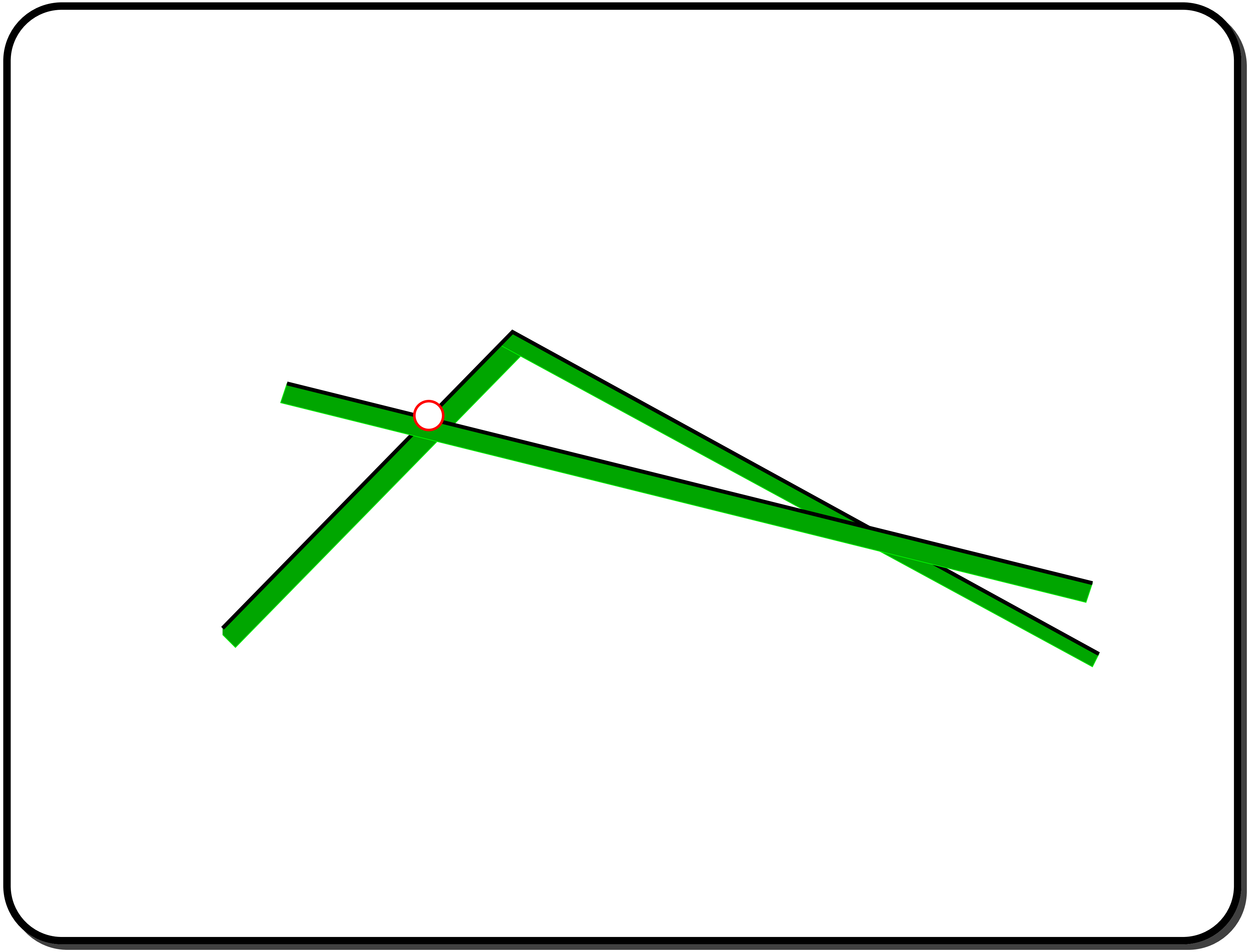
Conditional expected cost = (Unconditional) expected cost

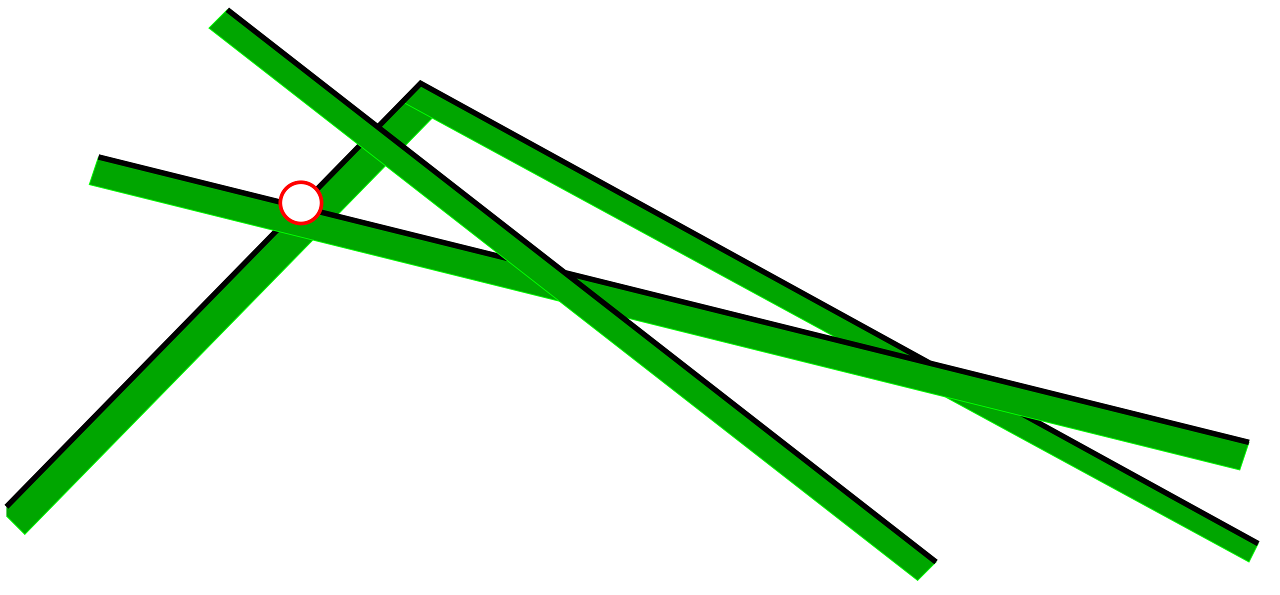
Linear programming (fixed Dim.)



Non degenerate : Exactly d constraints define the optimum.







Analysis

$\overline{T_d(n)}$ = Expected running time in d dimension for n constraints.

From backward analysis, probability that i-th insertion changes optimum is $\frac{d}{i}$ (d constraints define optimum).

$$\overline{T_d(i)} = \overline{T_d(i-1)} + \overline{T_{d-1}(i-1)} \cdot \frac{d}{i} + O(d)$$

By induction [Seidel]

$$\overline{T_d(n)} = O(d!n)$$

A generic search problem

Given a set $S \subset U$, build a data structure D, so that we can answer a query quickly.

Issues

- **query time**
- **space** for D (space)
- **Preprocessing time** to construct D

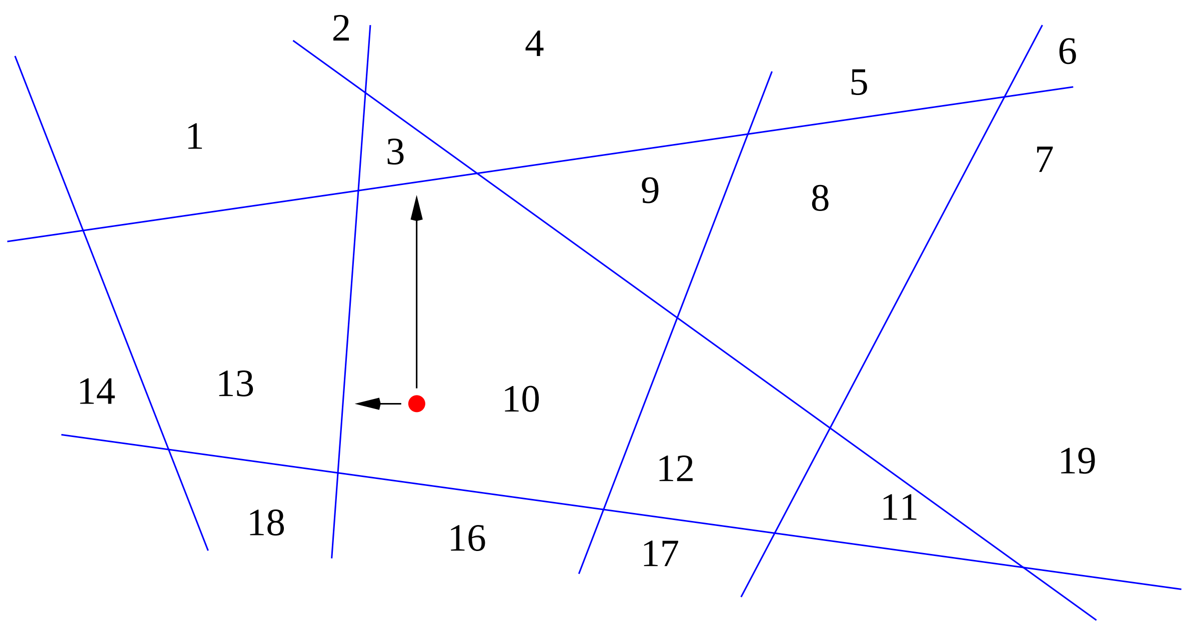
Dynamic version

- **Insert** update D for $S \cup x$, $x \in U - S$
- **Delete** update D for $S - x$

Ideal goal is to match the static performance and minimize update times.

Arrangement Searching

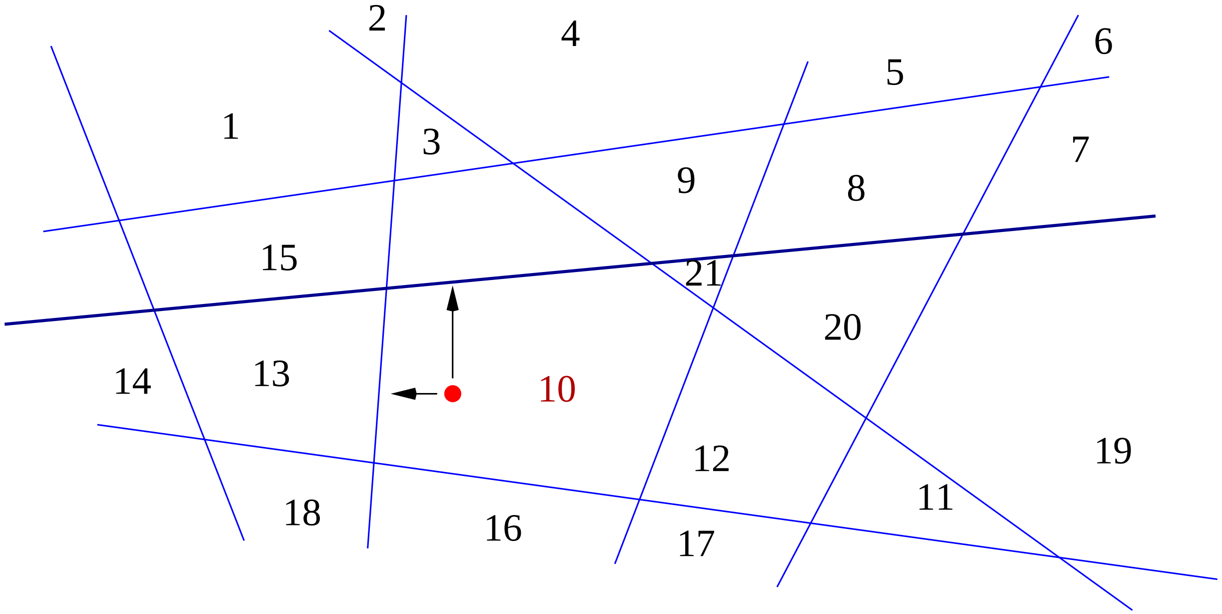
Problem : Given N lines(planes,hyperplanes), build data structure to do point location(report the face it lies in)



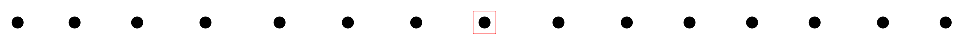
Arrangement Searching cont'd

Dynamic version :

Allow insertion/deletion of lines



Binary search



$$T(n) \leq T(\frac{n}{2}) + O(1)$$

Approximate split :

$$T(n) \leq T(\alpha n) + O(1)$$

where α is a constant < 1(independent of n)

Binary search cont'd

$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \boxed{\bullet} \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$$

Random split :

$$T(n) \leq T(x) + O(1)$$

where x is random variable uniformly distributed in $[1..n]$

$$\Pr[T(n) > c \log n] < \frac{1}{n}$$

Examples :
- Randomized search trees
- Quick sort

Review of randomized search

Simple binary search

$$T(n) = T(\frac{n}{2}) + O(1)$$

Approximate Split

$$T(n) \leq T(\alpha \cdot n) + O(1) \quad , \alpha < 1$$

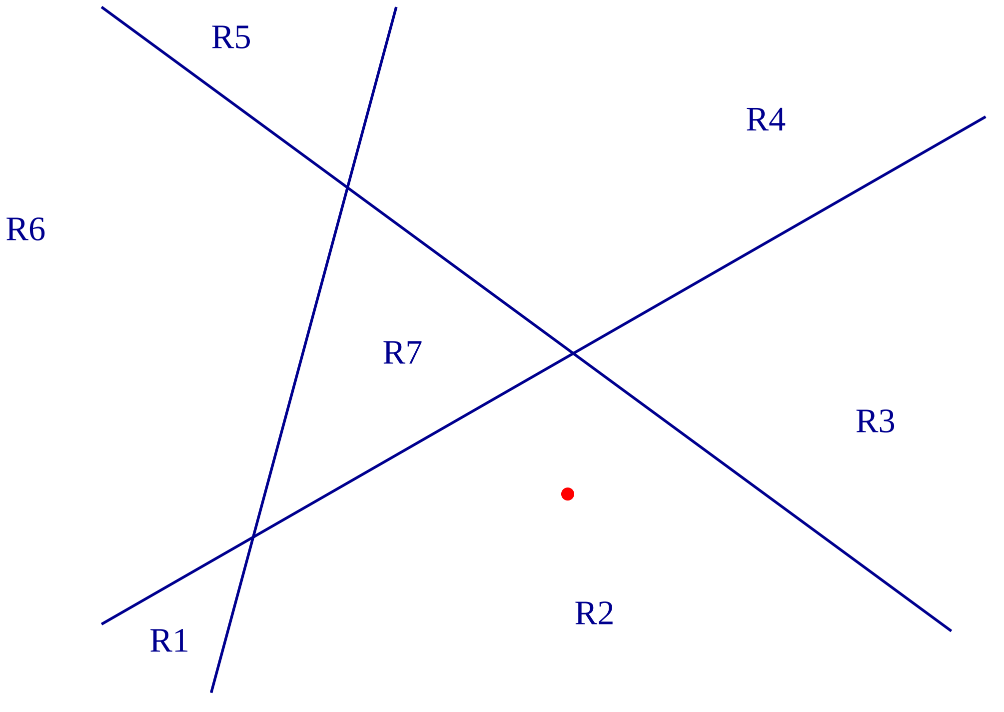
Random Split

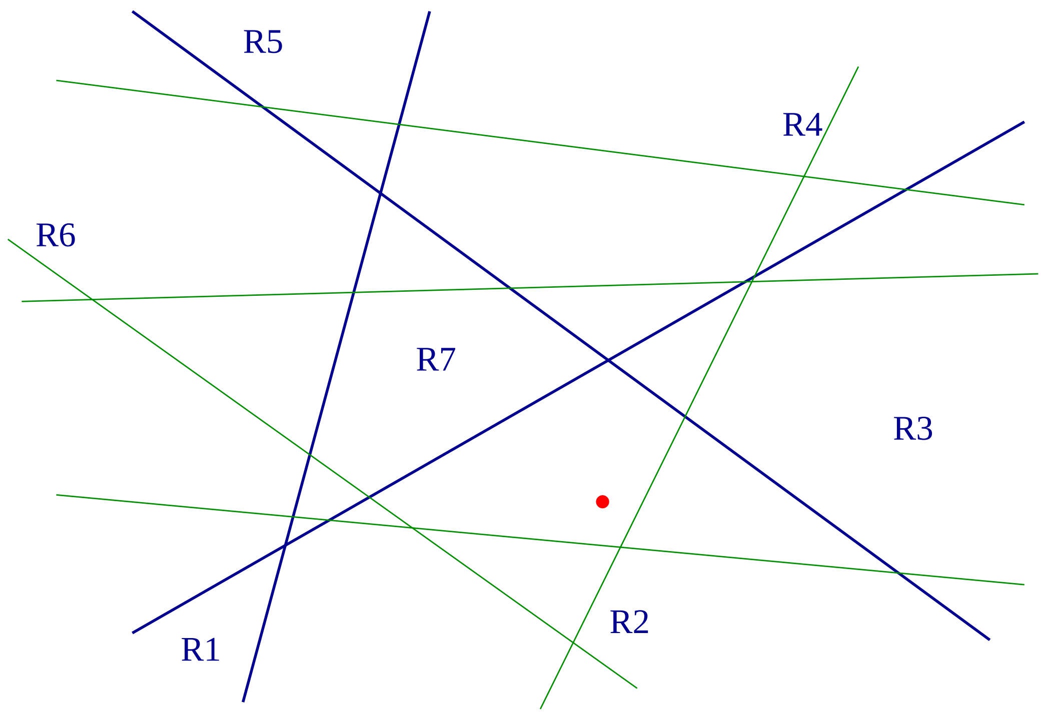
$$T(n) \leq T(X) + O(1)$$

X is a r.v. $\in [1 \ldots n]$

$$\Pr[T(n) > c \log n] < \frac{1}{n}$$

Randomized Search Trees/Quicksort


R5
R4
R6
R7
R3
R1
R2


R5
R4
R6
R7
R3
R2
R1

Generalizing the randomized search tree

- Choose a *good* sample R of size C.

- Split the input using R and build the data structure recursively for each subset.

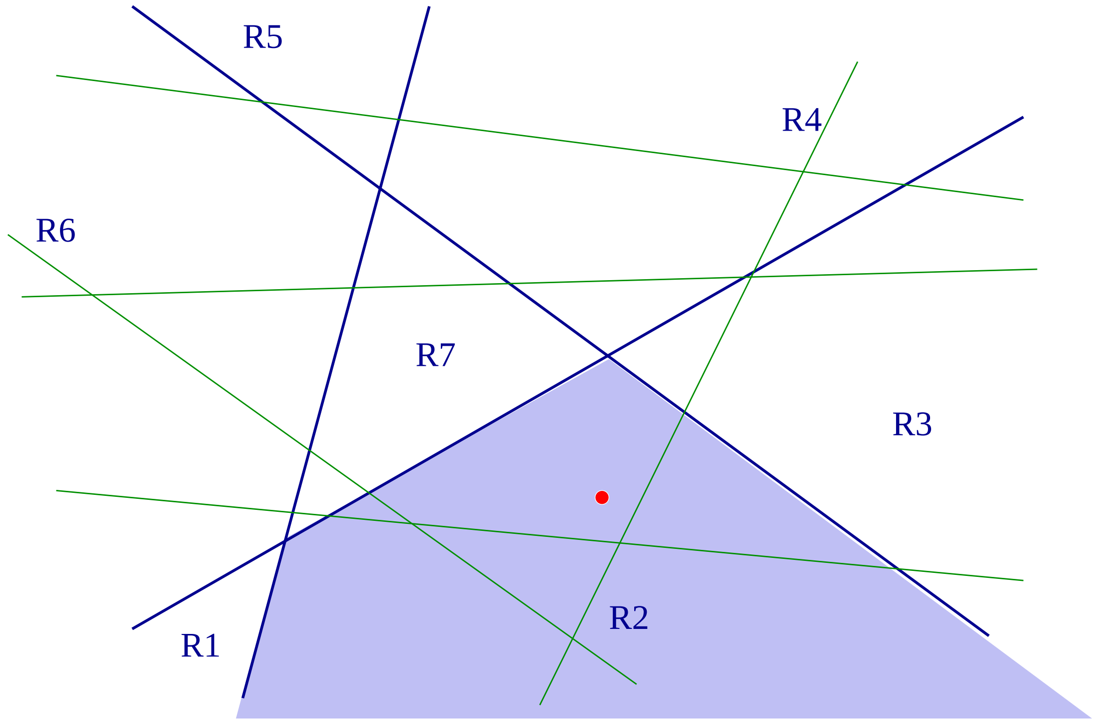
  R is good if each subproblem size is less than $n/2$

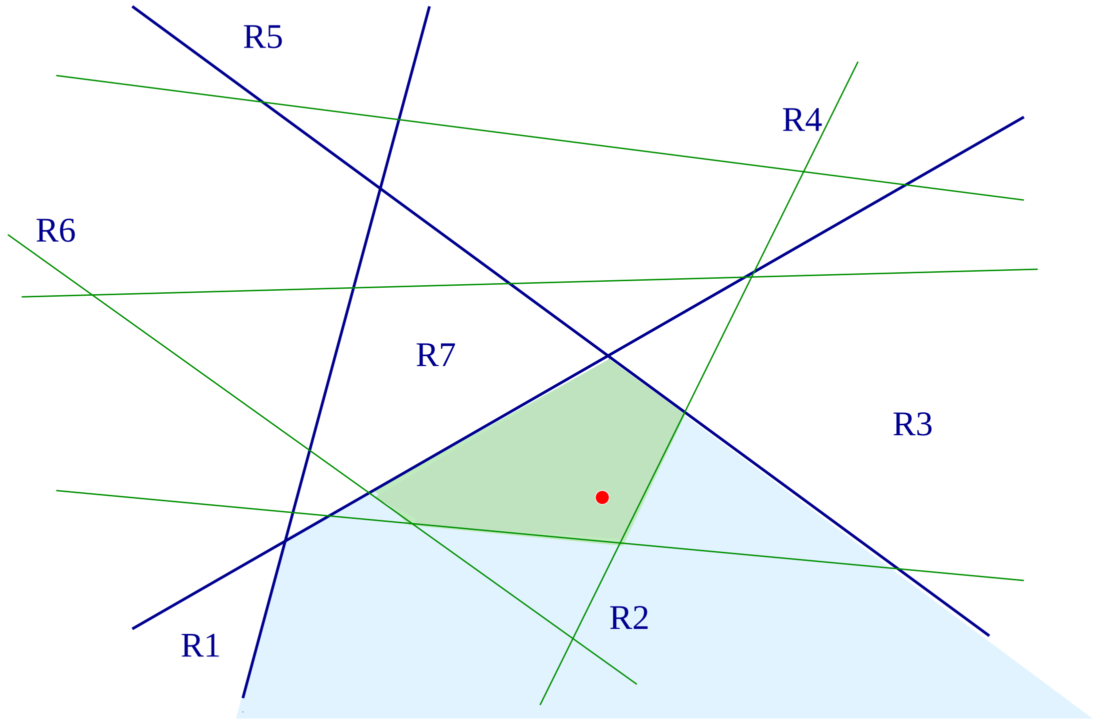
- C is large enough such that R is good with probability $\geq 1/2$, i.e. expected number of repetition ≤ 2.

- Height of data structure is $O(\log n)$, so search time is $O(\log n)$

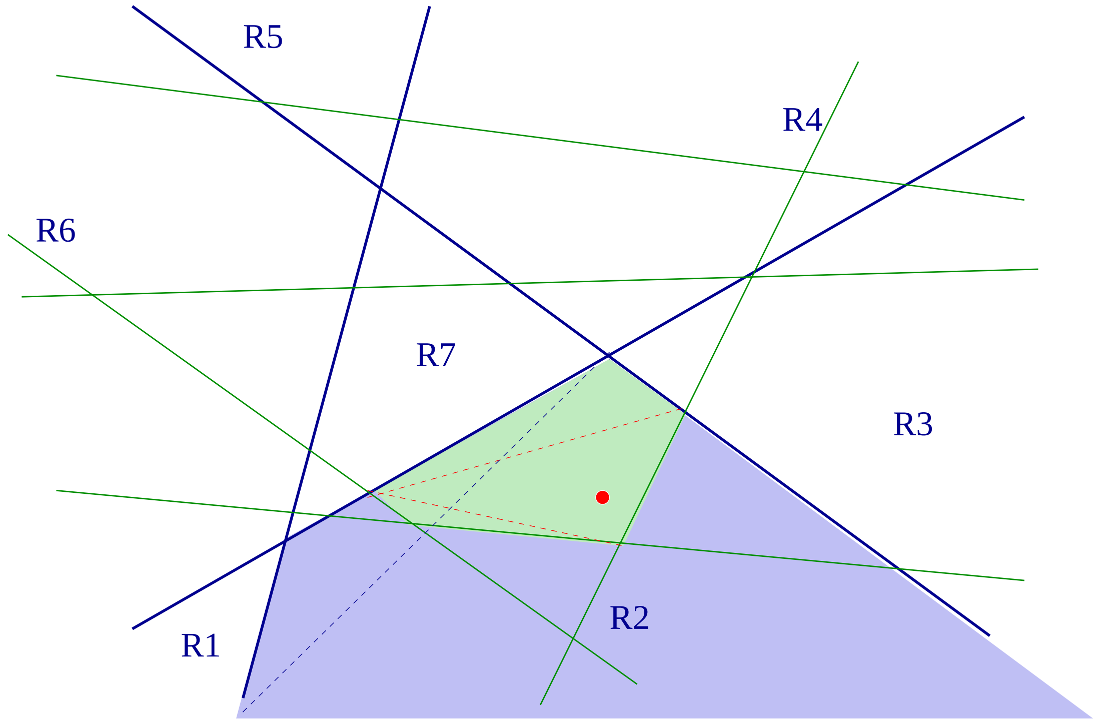
- **Space** Fragmentation makes it super-linear space. In this case $O\left(C^2 \frac{n}{C} \log C\right)$ or

$$O(nC \log C)$$


R5
R4
R6
R7
R3
R2
R1


R5
R4
R6
R7
R3
R2
R1

From triangles to triangles



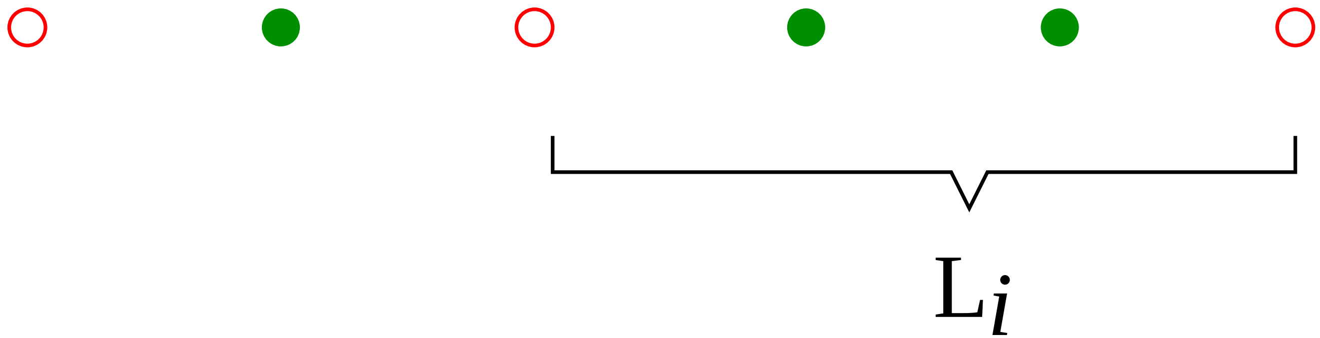
Reviewing skip list

Reviewing skip list cont'd

Reviewing skip list cont'd

A slightly different version

Choose an element to be in the sample with probbility 0.5



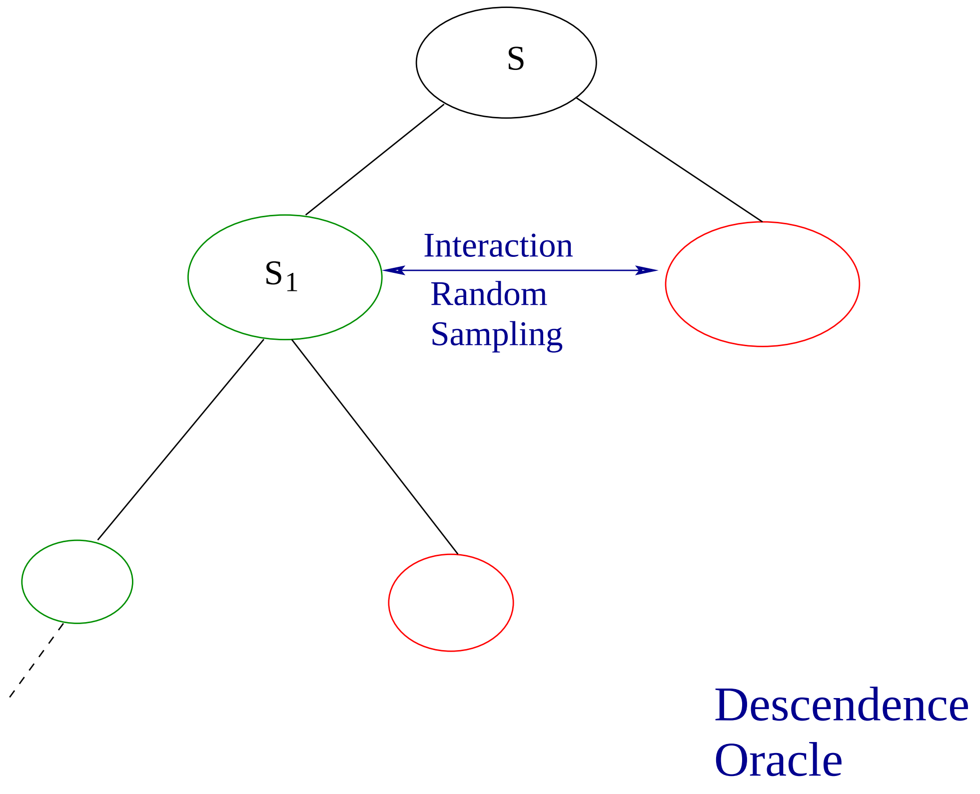
$$\text{Expectation}[L_i] = 2$$

PUGH Exp.[Total] $= O(\log n)$

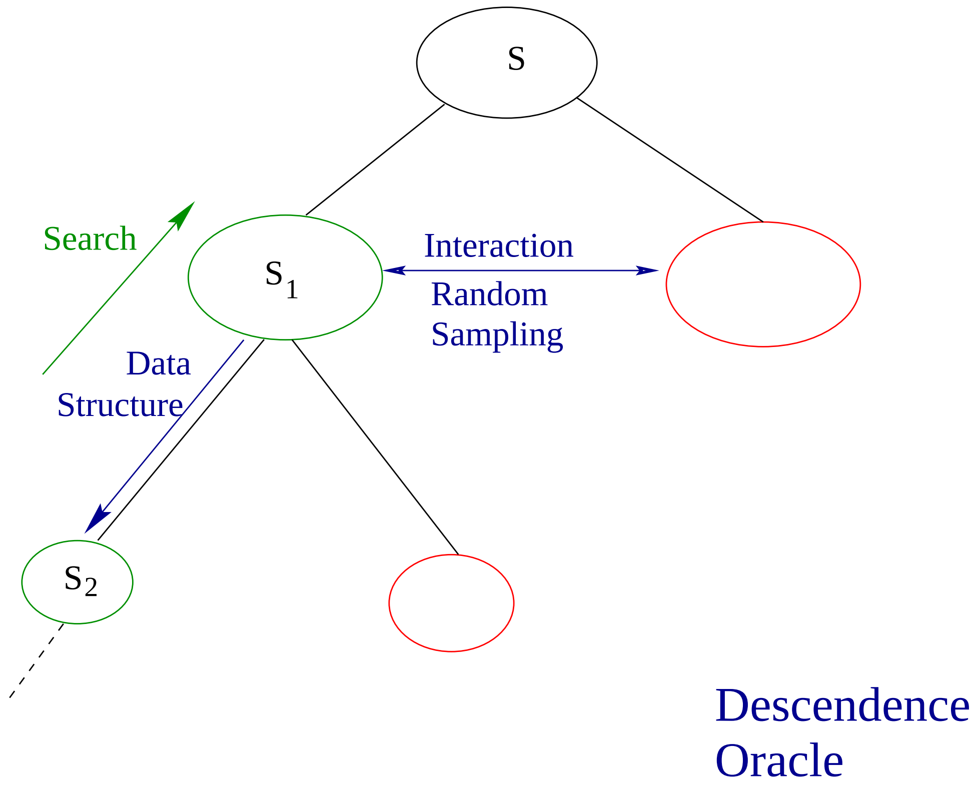
Improvement with careful analysis [Sen 91]:

$$\text{Total} < c \log n \text{ with probability } 1 - \frac{1}{n^c}$$

Overall structure

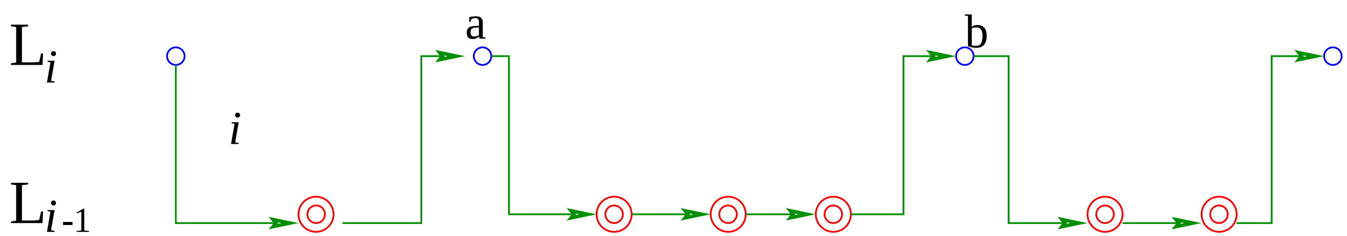


Overall structure



Descendence oracle for skip-lists

For each level L_i, maintain a linked-list of elements of L_{i-1} that an interval $[a, b]$, $a, b \in L_i$ intersects.



Descendence oracle for arrangement searching

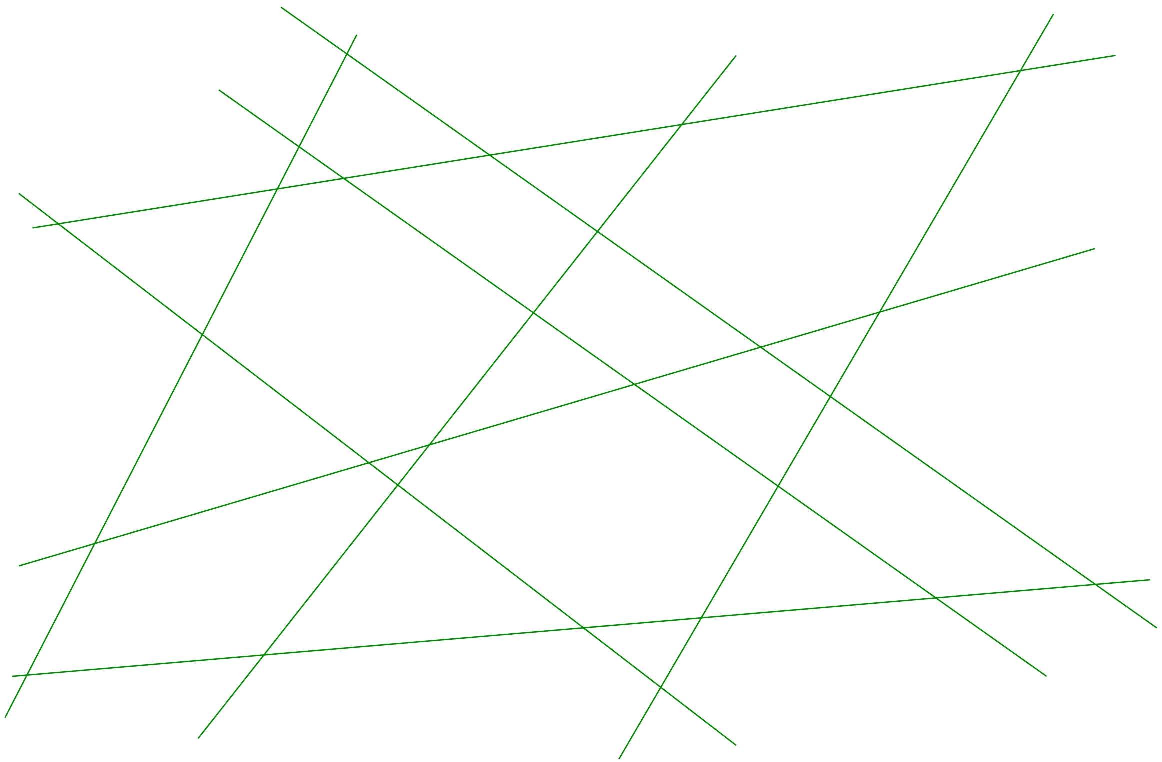
How many triangles of level i intersect a triangle of level $i-1$?

$$O\left(\frac{n}{r}\right) \quad \text{whp from Random sampling lemma}$$

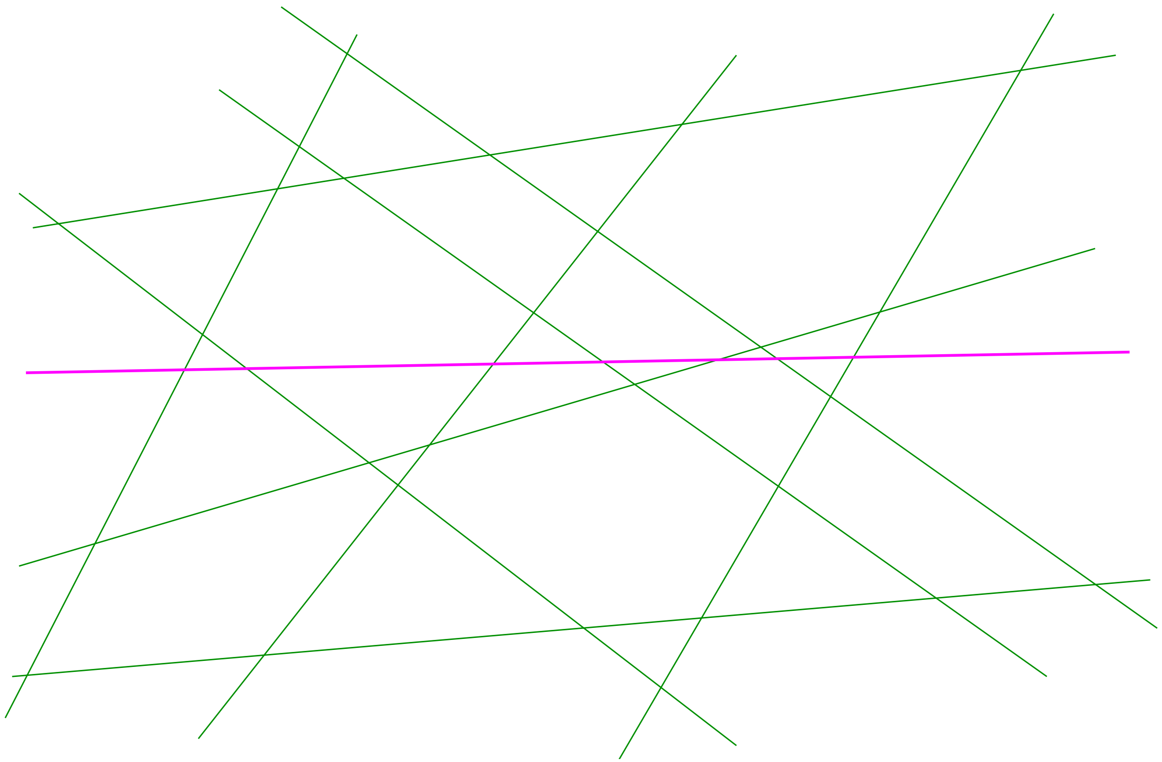
Descendence oracle can be a simple data structure storing the intersections between triangles of successive levels and takes time $O(\log n)$ w.h.p.

implying $O(\log^2 n)$ time w.h.p for overall query.

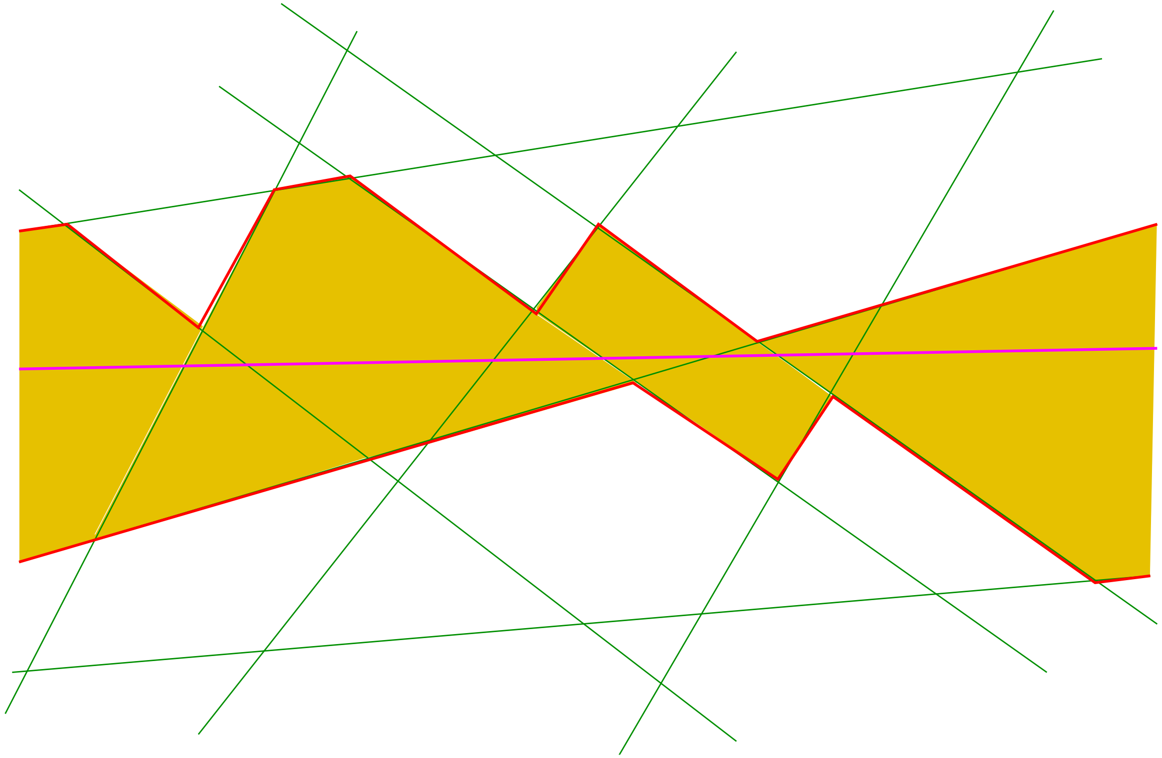
Updates



Update



<u>Zone lemma</u>



Size of zone is $O(n)$, i.e. only $O(n)$ triangles must be updates at each level.

Bounds

- **Searching** $O(\log n)$ w,h.p.
- **Update** $O(n \log n)$ w.h.p.
- **Space** $O(n^2)$

Dimension d (fixed)

- **Searching** $O(\log^{d-1} n)$ w.h.p.
- **Update** $O(n^{d-1} \log n)$ w.h.p.
- **Space** $O(n^d)$

[Mu-Sen]