

## Pumping Lemma for CFL

Let  $L$  be a CFL. Then there exists an integer  $n$ , s.t. for all  $s \in L$  and  $|s| \geq n$ ,  $s$  can be written as

$$s = u \cdot v \cdot w \cdot x \cdot y \quad \text{where}$$

$$|v| + |x| \geq 1 \quad |v \cdot w \cdot x| \leq n$$

$$\text{s.t. } \forall i \geq 0 \quad u v^i w x^i y \in L$$

Example:  $L = \{a^i b^i c^i, i \geq 1\}$

Suppose  $L$  is a CFL  $\Rightarrow$  P.L. can be applied

$$s = \underbrace{a \cdot a \cdots a}_n \quad \underbrace{b \cdot b \cdots b}_n \quad \underbrace{c \cdot c \cdots c}_n$$

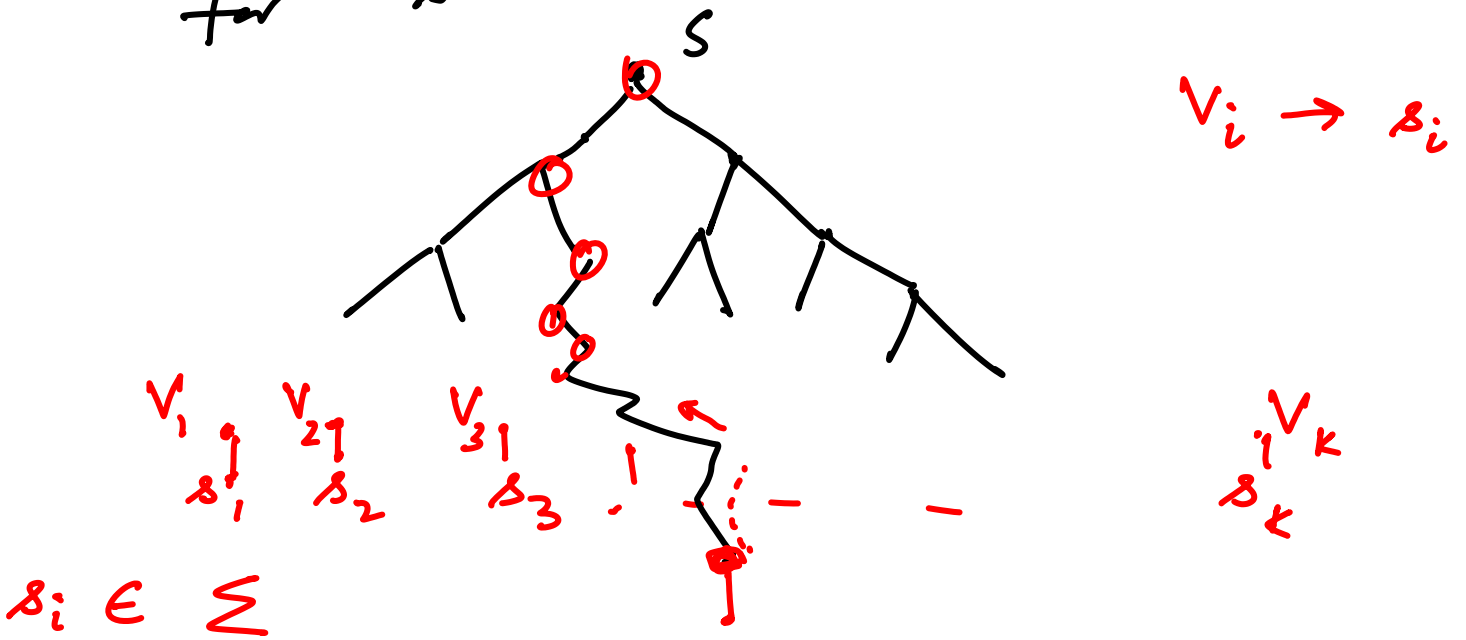
$u$        $v \quad w \quad x$        $y$

# Proof of PL for CFL

Assume that the CFL is in CNF

All prodn are  $A \rightarrow BC$  or

$S \in L$ . Consider a derivation tree for  $s$



In any binary tree with  $2^t$  leaf nodes there must be a path of length  $\geq t$

$\Rightarrow$  If  $t \geq |V|$  then some variable must be repeated in the derivation along the longest path

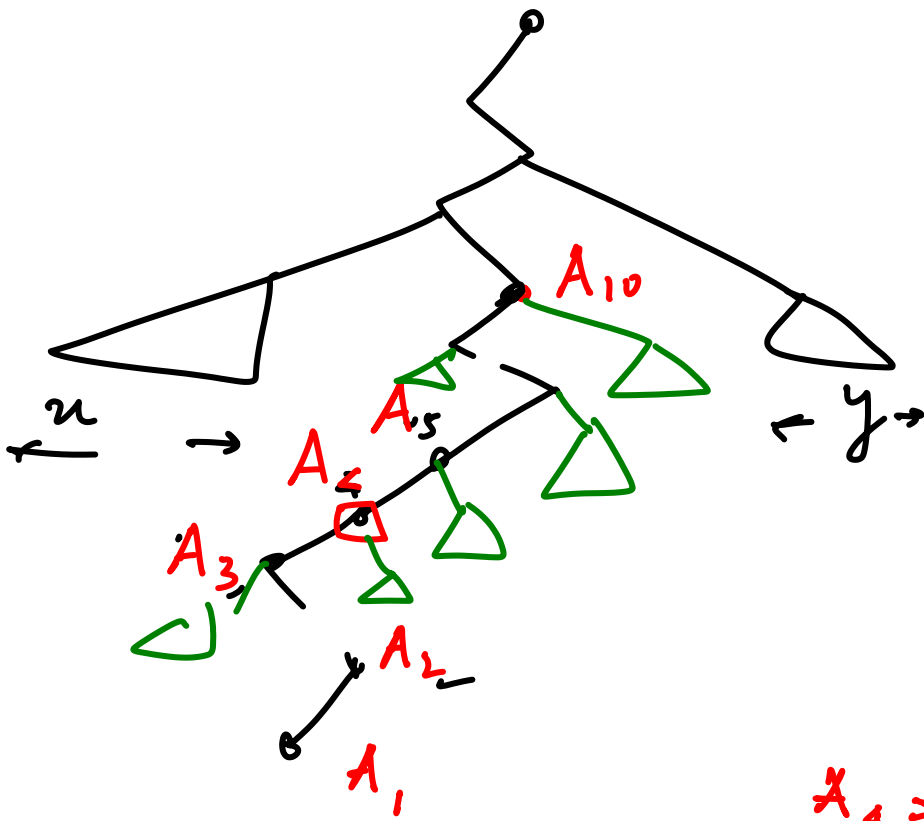
$$|V| = k$$

Path has length  $\geq k$

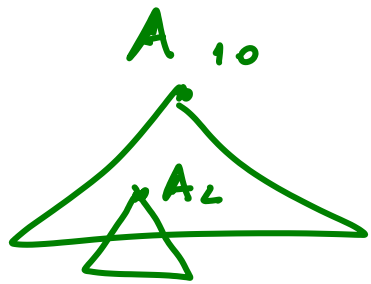
and so # of variables is  $\geq k+1$

$\Rightarrow$  Some variable is repeated

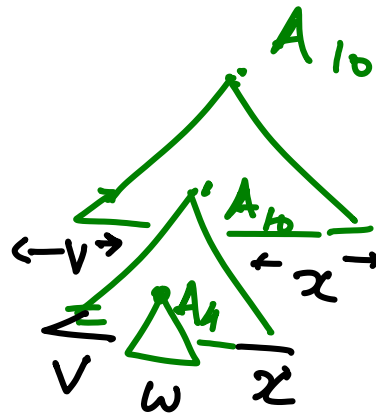
$$A_i \in V$$



$$A_4 = A_{10}$$



$\Rightarrow$



$$n = 2^k$$

$\uparrow$  P.L.  $n$

$$L = \{ ww \mid w \in (0+1)^* \}$$

$$L = \{ a^i b^j a^i b^j, i, j \geq 1 \}$$

$$a a b a a b \in L \quad a b a a b \notin L$$

Is  $L$  CFL?

Consider  $a^n b^n a^n b^n = u \left[ \overset{\leftarrow n}{v} \underset{\rightarrow}{w} x \right]$

Exercise Show  $L$  is not CFL using  
PL

Is CFL closed under union?

$L_1$  is a CFL, say  $S_1$  is the start symbol

$L_2$  is a CFL "  $S_2$  is the start symbol

Add the prodn  $S \rightarrow S_1 \mid S_2$

Keep the variable names disjoint  
between  $L_1$  and  $L_2$

If  $L_1$  is CFL and  $L_2$  CFL  
is  $L_1 \cap L_2$  CFL?

Can we do a product construction  
of two PDAs,  $M_1, M_2$ ?

$$Q = Q_1 \times Q_2$$

Can we simulate 2 stacks by 1 stack?

$$L_1 = \{ a^i b^i c^j \mid i, j \geq 1 \}$$

$$L_2 = \{ a^j b^i c^i \mid i, j \geq 1 \}$$

$$L = L_1 \cap L_2 = \{ a^i b^i c^i \mid i \geq 1 \}$$

$L$  is not a CFL, whereas  $L_1$  and  $L_2$  are CFL  
CFLs are not closed under intersection

Corollary CFL are not closed  
under complementation

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

Special case :  $L_1$  : CFL  
 $L_2$  : Regular

$L = L_1 \cap L_2$  is a CFL.

Use product construction : only  
one stack required.

$$L = \{ ww \mid w \in (0+1)^* \}$$

Is  $L$  CFL?

Suppose  $L$  is a CFL

Consider  $L_1 = \{ a^i b^j a^i b^j, i, j \geq 1 \}$

$L_2 = a^+ b^+ a^+ b^+$   
is regular

$$L \cap L_2 = L_1$$

Contradiction since  $L_1$  is not CFL

### Decision problems on CFL

1. Given a CFL is it

(i) finite      (ii) infinite      (iii)  $\emptyset$