COL866: Quantum Computation and Information

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Quantum Computation: Phase estimation

Phase estimation

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- We will use the assumption that there are black-boxes that:
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  - perform the controlled-U^{2^j} operation.
- We will describe a phase estimation procedure that uses two registers:
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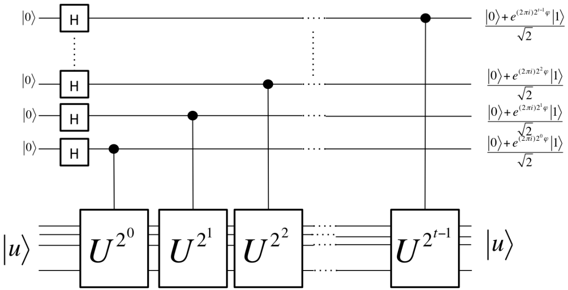

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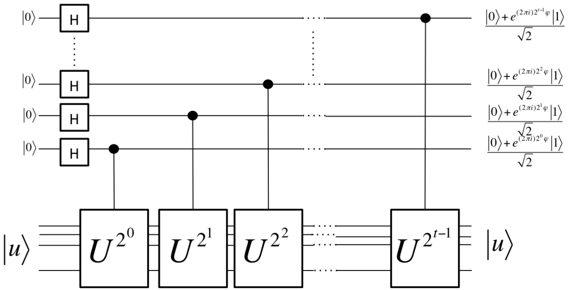

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- Let $0 \leq b \leq 2^t - 1$ be an integer such that $\frac{b}{2^t} = [0 \cdot b_1 ... b_t]$ is the best t bit approximation to φ that is less than φ. Let $\delta = \varphi - \frac{b}{2^t}$ (which implies $0 \leq \delta \leq 2^{-t}$).
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- Claim 2.2: Let α_l be the amplitude of $|(b + l) \ mod \ 2^t\rangle$. Then $\alpha_l = \frac{1}{2^t} \left(\frac{1 - e^{(2\pi i)(2^t \varphi - (b+l))}}{1 - e^{(2\pi i)(\varphi - (b+l)/2^t)}} \right) = \frac{1}{2^t} \left(\frac{1 - e^{(2\pi i)(2^t \delta - l)}}{1 - e^{(2\pi i)(\delta - l/2^t)}} \right)$.

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- The claim follows by setting $t = n + p$ and $\varepsilon = \frac{1}{2(2^p - 1)}$. $\square$

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- The phase estimation protocol works when the second register is set to the eigenstate $|u\rangle$. In general, this may not be feasible.
- Observation: Any general state $|\psi\rangle$ may be written in terms of the eigenstates of U as $\sum_u c_u |u\rangle$.
- Exercise: The phase estimation procedure takes state $(|0\rangle)(\sum_u c_u |u\rangle)$ to $\sum_u c_u |\tilde{\varphi_u}\rangle |u\rangle$. If $t = n + \lceil \log\left(2 + \frac{1}{2\varepsilon}\right) \rceil$, then the probability of measuring φ_u accurate to n bits at the end of the phase estimation procedure is at least $|c_u|^2(1 - \varepsilon)$.

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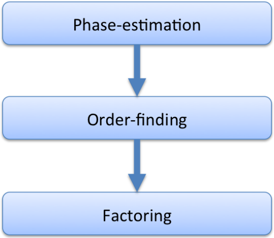
- Phase estimation enables us to design quantum algorithms for the order-finding and factoring problems.



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