COL866: Quantum Computation and Information

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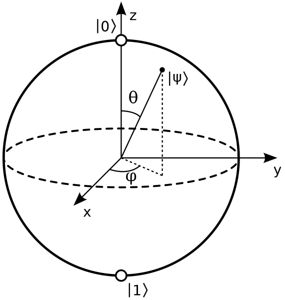
Quantum Computation: Quantum circuits

- Single qubit gates are 2×2 unitary matrices. Some of the important gates are:
  - Pauli matrices: $X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
  - Hadamard gate: $H \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.
  - Phase gate: $S \equiv \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$.
  - $\pi/8$ gate: $T \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$
- Simplification: A qubit $\alpha |0\rangle + \beta |1\rangle$ may be represented as $\cos\frac{\theta}{2} |0\rangle + e^{i\psi} \sin\frac{\theta}{2} |1\rangle$. So, any tuple (θ, ψ) represents a qubit.
- This has a nice visualization in terms of Bloch sphere.



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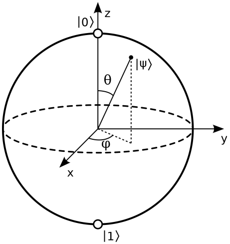

- The vector $(\cos \psi \sin \theta, \sin \psi \sin \theta, \cos \theta)$ is called the Bloch vector.

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- Pauli matrices give rise to three useful classes of unitary matrices when they are exponentiated, the rotational operators about the $\hat{x}$, $\hat{y}$, and $\hat{z}$ axis.

$$R_x(\theta) \equiv e^{-i\theta X/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} X = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_y(\theta) \equiv e^{-i\theta Y/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Y = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

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- A few useful results:
  - Let $\hat{n} = (n_x, n_y, n_z)$ be a real unit vector. The rotation by θ about the $\hat{n}$ axis is given by

$$R_{\hat{n}}(\theta) \equiv e^{-i\frac{\theta}{2}(\hat{n}\cdot\vec{\sigma})} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2}(n_x X + n_y Y + n_z Z),$$

    where $\vec{\sigma}$ denotes the vector (X, Y, Z).
  - Theorem: Suppose U is a unitary operator on a single qubit. Then there exist real numbers α, β, γ, and δ such that $U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$.

Theorem

Suppose U is a unitary operator on a single qubit. Then there exist real numbers α, β, γ, and δ such that $U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$.

Proof sketch

There are real numbers $\alpha, \beta, \gamma, \delta$ such that:

$$U = \begin{bmatrix} e^{i(\alpha - \beta/2 - \delta/2)} \cos \frac{\gamma}{2} & -e^{i(\alpha - \beta/2 + \delta/2)} \sin \frac{\gamma}{2} \\ e^{i(\alpha + \beta/2 - \delta/2)} \sin \frac{\gamma}{2} & e^{i(\alpha + \beta/2 + \delta/2)} \cos \frac{\gamma}{2} \end{bmatrix}$$

Now one just needs to verify that the RHS matches $e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$.

Theorem

Suppose U is a unitary operator on a single qubit. Then there exist real numbers α, β, γ, and δ such that $U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$.

Theoerm

Suppose U is a unitary gate on a single qubit. Then there exist unitary operators A, B, C on a single qubit such that $ABC = I$ and $U = e^{i\alpha} AXBXC$, where α is some overall phase factor.

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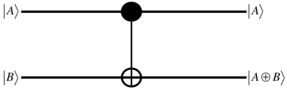
- Summary:
  - The above matrices are fundamental entities that define general classes of single-qubit unitary gates such that any single-qubit unitary gate can be represented in terms of these gates.

- The simplest two-qubit gate is the Controlled-NOT or `CNOT` gate:



with matrix representation $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. The top qubit is called the control qubit, and the bottom qubit is called the target qubit.

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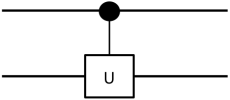

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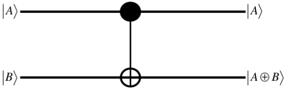

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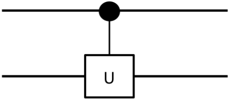

- Some exercises:
  - Build a `CNOT` gate from one Controlled-Z gate and two Hadamard gates.

Quantum Circuit
Controlled operations

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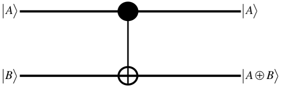

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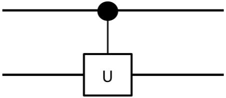

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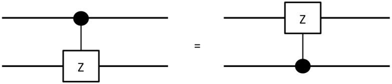

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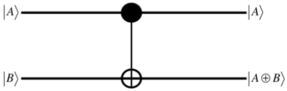

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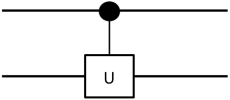

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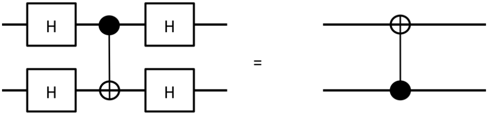

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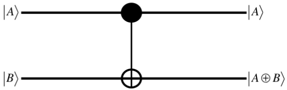

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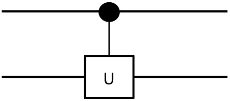

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For a single qubit U, can we implement Controlled-U gate using only `CNOT` and single-qubit gates?

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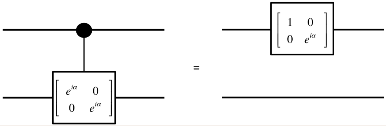
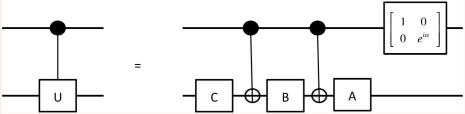
Construction sketch

The construction follows from the following circuit equivalences.





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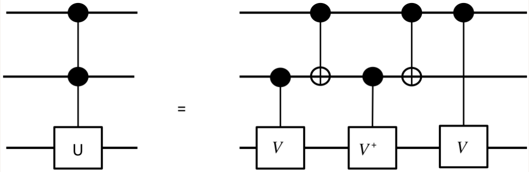

Here V is such that $V^2 = U$.

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For a single qubit U, can we implement Controlled-U gate with n control qubits using only `CNOT` and single-qubit gates?

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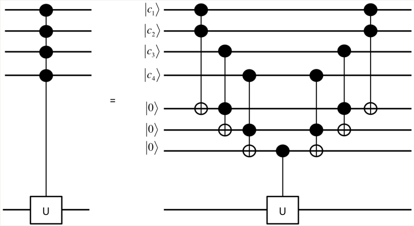
For a single qubit U, can we implement Controlled-U gate with two control qubits using only CNOT and single-qubit gates? Yes

Question

For a single qubit U, can we implement Controlled-U gate with n control qubits using only CNOT and single-qubit gates? Yes using ancilla qubits

Construction sketch

An example construction with $n = 4$.



Quantum Circuit
Controlled operations

- A few other gates and circuit identities:

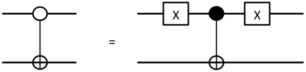
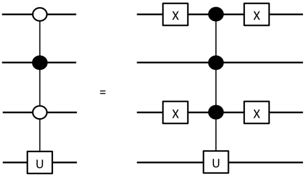
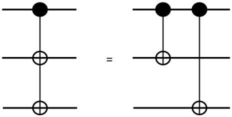


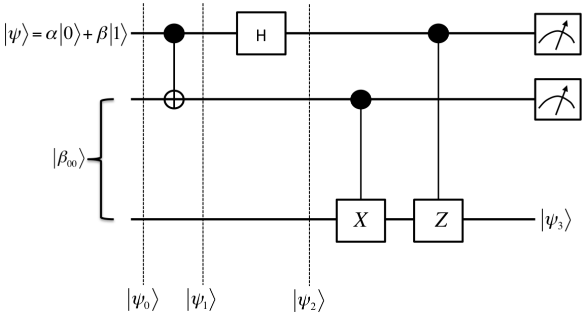
Figure: `NOT` gate applied to the target qubit conditional on the control qubit being 0.





Principle of deferred measurements

Measurements can always be moved from an intermediate stage of a quantum circuit to the end of the circuit; if the measurement results are used at any stage of the circuit, then the classically controlled operations can be replaced by conditional quantum operations.



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Principle of implicit measurement

Without loss of generality, any unterminated quantum wires (qubits that are not measured) at the end of a quantum circuit may be assumed to be measured.

- A set of gates is said to be universal for quantum computation if **any** unitary operation may be **approximated** to arbitrary accuracy by a quantum circuit involving only those gates.

Claim

Any unitary operation can be approximated to arbitrary accuracy using Hadamard, `CNOT`, and $\pi/8$ gates.

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Proof sketch

- Claim 1: A single qubit operation may be **approximated** to arbitrary accuracy using the Hadamard, and $\pi/8$ gates.
- Claim 2: An arbitrary unitary operator may be expressed **exactly** using single qubit and CNOT gates.
    - Claim 2.1: An arbitrary unitary operator may be expressed **exactly** as a product of unitary operators that each acts non-trivially only on a subspace spanned by two computational basis states (such gates are called two-level gates).
    - Claim 2.2: An arbitrary two-level unitary operator may be expressed exactly using single qubit and CNOT gates.

- What about efficiency?
    - Upper-bound: Any unitary can be approximated using exponentially many gates.
    - Lower-bound: There exists a unitary operation that requires exponentially many gates to approximate.

Claim 2.1

An arbitrary unitary operator may be expressed **exactly** as a product of unitary operators that each acts non-trivially only on a subspace spanned by two computational basis states.

Proof sketch

- The main idea can be understood using a 3×3 unitary matrix:

$$U = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & j \end{bmatrix}.$$

- We will find two-level unitary matrices U_1, U_2, U_3 such that

$$U_3 U_2 U_1 U = I \quad \text{and} \quad U = U_1^\dagger U_2^\dagger U_3^\dagger$$

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- Exercise
  - Show that any $d \times d$ unitary matrix can be written in terms of $d(d-1)/2$ two-level matrices.
  - There exists a $d \times d$ unitary matrix U which cannot be decomposed as a product of fewer than $d-1$ two-level unitary matrices.

Claim 2

An arbitrary unitary operator may be expressed **exactly** using single qubit and CNOT gates.

- Claim 2.1: An arbitrary unitary operator may be expressed **exactly** as a product of unitary operators that each acts non-trivially only on a subspace spanned by two computational basis states.
- Claim 2.2: An arbitrary two-level unitary operator may be expressed exactly using single qubit and CNOT gates.

Proof sketch

- Let U be a two-level unitary matrix on a n-qubit quantum computer.
- Let U act non-trivially on the space spanned by the computational basis states $|s\rangle$ and $|t\rangle$, where $s = s_1, ..., s_n$ and $t = t_1, ..., t_n$ are n-bit binary strings.
- Let $\tilde{U}$ be the non-trivial 2×2 submatrix of U. Note that we can think $\tilde{U}$ to be a unitary operator on a single qubit.
- We will use the gray-code connecting s and t, which is a sequence of n-bit strings starting with s and ending with t such that the subsequent strings in the sequence differ only on one bit.
- Example: $s = 101001$, $t = 110011$.

$$g_1 = 101001; g_2 = 101011; g_3 = 100011; g_4 = 110011$$

- Main idea:
  - We will design a sequence of swaps $|g_1\rangle \rightarrow |g_{m-1}\rangle, |g_2\rangle \rightarrow |g_1\rangle, |g_3\rangle \rightarrow |g_2\rangle, ..., |g_{m-1}\rangle \rightarrow |g_{m-2}\rangle$.
  - We will apply $\tilde{U}$ to the qubit that differs in g_{m-1} and g_m.
  - Swap $|g_{m-1}\rangle$ with $|g_{m-2}\rangle$, $|g_{m-2}\rangle$ with $|g_{m-3}\rangle$ and so on.

Claim 2.2

An arbitrary two-level unitary operator may be expressed exactly using single qubit and `CNOT` gates.

Example construction

- Let the two-level transformation be:

$$U = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & d \end{bmatrix}$$

- The gray code connecting $|000\rangle$ and $|111\rangle$:
$|000\rangle \rightarrow |001\rangle \rightarrow |011\rangle \rightarrow |111\rangle$.

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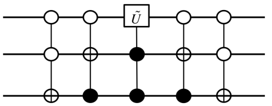

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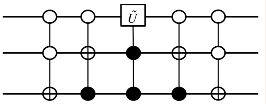

- Exercise
  - For an arbitrary unitary operator on an n-qubit system, how many CNOT and single qubit gate will be required in the entire construction?

Claim 2

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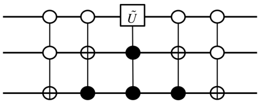
Example construction

- Let the two-level transformation be:

$$U = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & d \end{bmatrix}$$

- The gray code connecting $|000\rangle$ and $|111\rangle$: $|000\rangle \rightarrow |001\rangle \rightarrow |011\rangle \rightarrow |111\rangle$.
- Construction:



- Exercise
  - For an arbitrary unitary operator on an n-qubit system, how many `CNOT` and single qubit gate will be required in the entire construction? $O(n^2 4^n)$ gates.

End