COL866: Quantum Computation and Information

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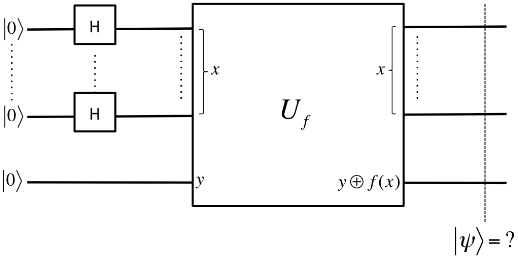
Introduction

- Question: Can we generalize this idea for boolean functions over multiple bit inputs?
- Consider any boolean function over n-bit inputs $f : \{0,1\}^n \rightarrow \{0,1\}$.
- What is the output of the following circuit?



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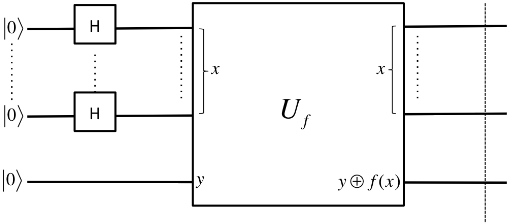

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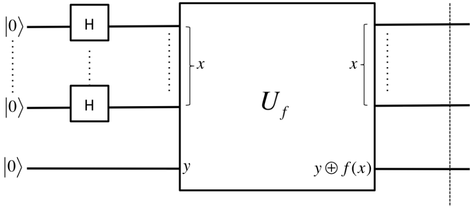

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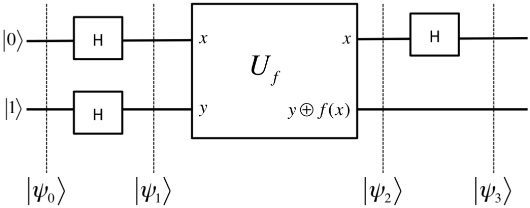
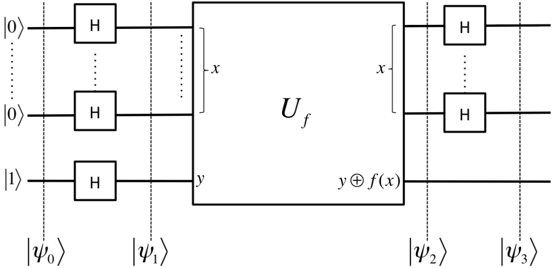
- Exploiting the parallelism in more realistic way is the key challenge while designing quantum algorithms.
- Consider the case of a boolean function on single-bit inputs $f : \{0,1\} \rightarrow \{0,1\}$. Suppose we would want to know if $f(0) = f(1)$. Here is a quantum circuit that solves this.

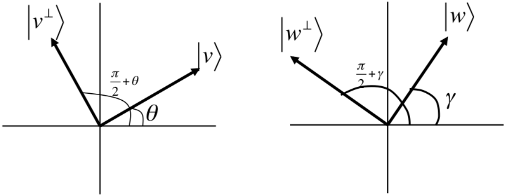


- The previous problem was a specific case of the more general Deutsch's problem that further demonstrates the power of quantum algorithms.
- Deutsch's problem: Bob has a function $f : \{0,1\}^n \rightarrow \{0,1\}$ that is either a constant function or a balanced function (i.e., f is 0 on $2^n/2$ inputs). Alice wants to determine what kind of function Bob has but can make a query to the function only once.
- The following circuit does this:



Introduction: Entanglement

- We said that one can measure in any orthonormal basis.
- Often, we would want to measure in a basis that is rotation of the standard basis.



- So, $|v\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$ and $|v^{\perp}\rangle = -\sin\theta\,|0\rangle + \cos\theta\,|1\rangle$
- Claim: Making a measurement in the $\{|v\rangle, |v^{\perp}\rangle\}$ basis is the same as making a measurement in the standard basis after applying the following gate:

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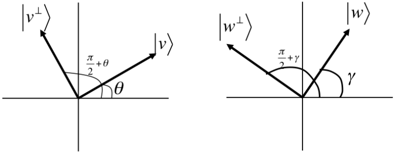

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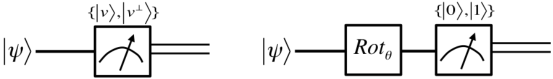
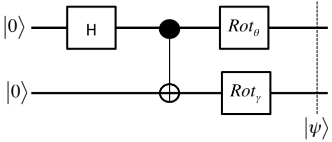
- Claim: Making a measurement in the $\{|v\rangle, |v^{\perp}\rangle\}$ basis is the same as making a measurement in the standard basis after applying the following gate: $Rot_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.
- In terms of circuits, the following two circuits exhibit the same measurement results.



- Let $\Delta = \theta - \gamma$. What is output of the following circuit $|\psi\rangle$?



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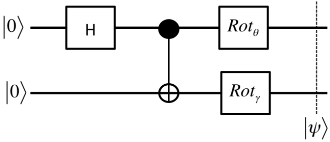
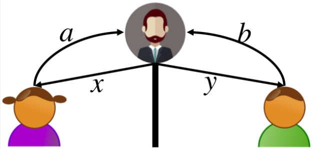


$|\psi\rangle = \frac{1}{\sqrt{2}} (\cos\Delta\,|00\rangle + \sin\Delta\,|01\rangle - \sin\Delta\,|10\rangle + \cos\Delta\,|11\rangle)$

- Corollary: Suppose Alice has the first qubit and Bob has the second qubit. Then on measurement of $|\psi\rangle$, the output is same with probability $\cos^2\Delta$ and different with probabiity $\sin^2\Delta$.

CHSH game

Alice and Bob receive randomly generated bits $x, y \in \{0, 1\}$ respectively from a Charlie. Their goal is to respond with bits a and b such that $a \oplus b = x \wedge y$. They are not allowed to communicate after receiving x and y.



- Lemma 1: There is no classical deterministic or randomized strategy that allows Alice and Bob to win with probability more than $3/4$.
- Lemma 2: There is a quantum strategy that allows Alice and Bob to win with probability $\cos^2 \pi/8 \approx 0.85 > 3/4$.

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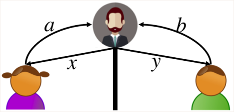

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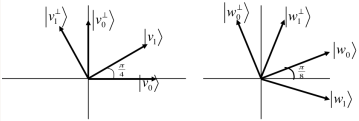
Quantum strategy

- Alice and Bob share an EPR pair $\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$ to start with.
- Alice and Bob measure in basis $\{|v_x\rangle, |v_x^{\perp}\rangle\}$, $\{|w_x\rangle, |w_x^{\perp}\rangle\}$ respectively and they simply return their measurement outputs.



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