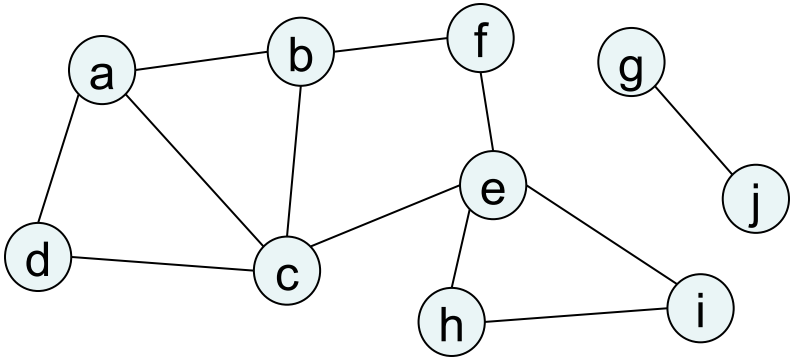
COL702:Advanced Data Structures and Algorithms

Thanks to Miles Jones, Russell Impagliazzo, and Sanjoy Dasgupta at UCSD for these slides.

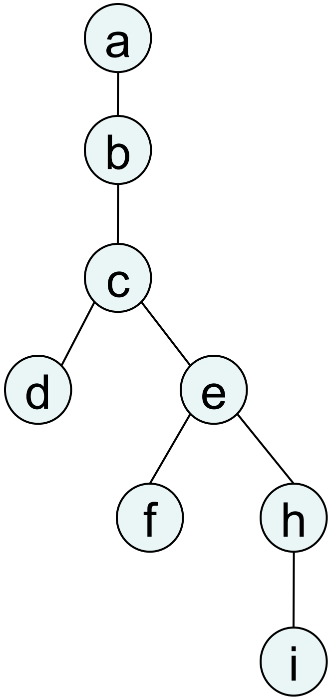
Paths in graphs

The classic 15-puzzle

Graph G = (V,E)
V = {configurations of puzzle}
E: edges between neighboring configurations



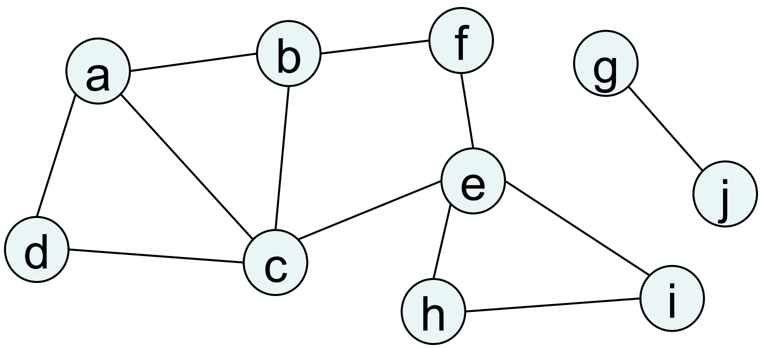
explore(G,a):



Finds a path from a to i.
But this isn't the shortest possible path!

Distances in graphs

Distance between two nodes
= length of shortest path between them

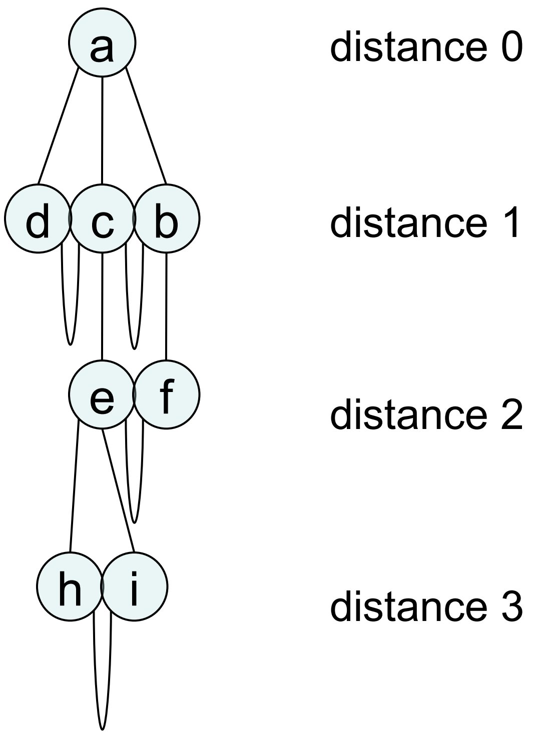


dist(a,e) = ?
dist(d,g) = ?

Suppose we want to compute distances from some starting node s to all other nodes in G.

Strategy: layer-by-layer
- first, nodes at distance 0
- then, nodes at distance 1
- then, nodes at distance 2, etc.

Physical model:
Vertex – ping-pong ball
Edge – piece of string



Breadth-first search

Suppose we have seen all nodes at distance $\leq$ d.

How to get the next layer?

Solution:

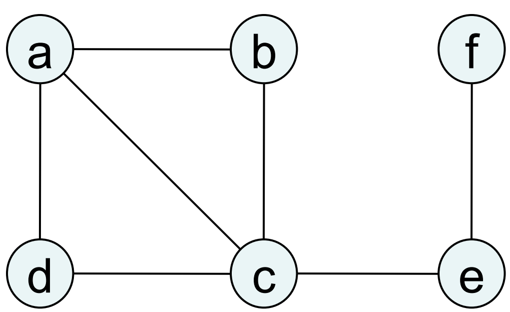
A node is at distance d+1 if:

- it is adjacent to some node at distance d
- it hasn't been seen yet

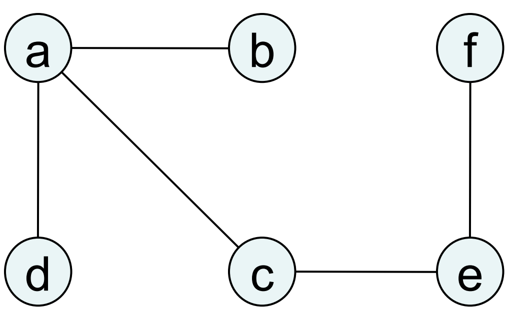
```
procedure bfs(G,s)

input: graph G = (V,E); node s in V
output: for each node u, dist[u] is
   set to its distance from s

for u in V:
   dist[u] = ∞
dist[s] = 0
Q = [s]  // queue containing just s

while Q is not empty:
   u = eject(Q)
   for each edge (u,v) in E:
        if dist[v] = ∞:
              inject(Q,v)
              dist[v] = dist[u]+1
```

BFS example

```
procedure bfs(G,s)

for u in V:
   dist[u] = ∞
   prev[u] = nil
dist[s] = 0
Q = [s]  // queue containing just s
while Q is not empty:
   u = eject(Q)
   for each edge (u,v) in E:
        if dist[v] = ∞:
                inject(Q,v)
                dist[v] = dist[u]+1
                prev[v] = u
```



Queue	Distances					
	a	b	c	d	e	f
[a]	0	∞	∞	∞	∞	∞
[bcd]	0	1	1	1	∞	∞
[cd]	0	1	1	1	∞	∞
[de]	0	1	1	1	2	∞
[e]	0	1	1	1	2	∞
[f]	0	1	1	1	2	3
[]	0	1	1	1	2	3



Shortest path tree

Why does BFS work?

```
procedure bfs(G,s)

for u in V:
    dist[u] = ∞
dist[s] = 0
Q = [s]
while Q is not empty:
    u = eject(Q)
    for each edge (u,v) in E:
        if dist[v] = ∞:
            inject(Q,v)
            dist[v] = dist[u]+1
```

Claim For any distance d = 0,1,2,..., there is a point in time at which:

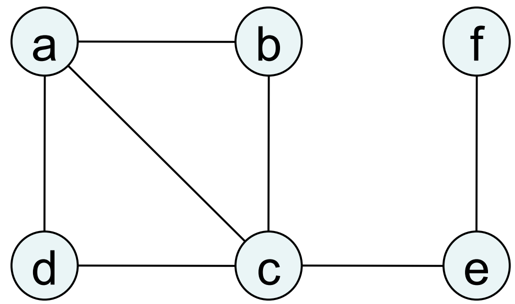
(i) all nodes at distance $\leq$ d have their dist[] values correctly set

(ii) all other nodes have dist[] = ∞

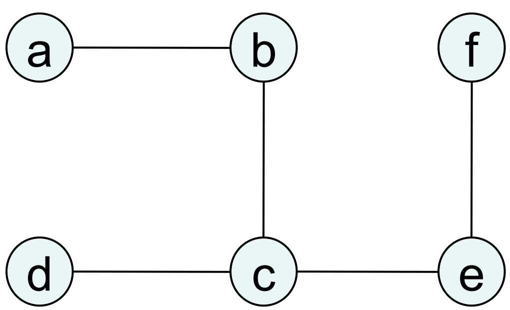
(iii) the queue Q contains exactly the nodes at distance d

Running time: O(V + E), like DFS

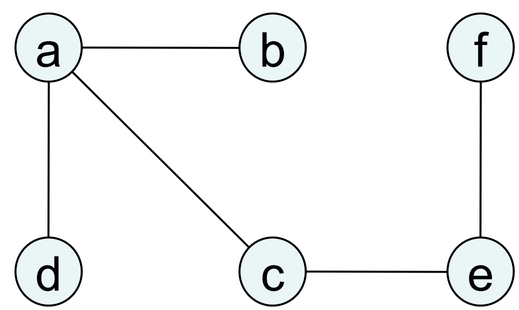
Two search strategies



Depth-first



Breadth-first



Edge lengths

BFS treats all edges as having the same length.
This is rarely true in applications.

Denote the length of edge e = (u,v) by l(e) or l_e or l(u,v)

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Edge lengths

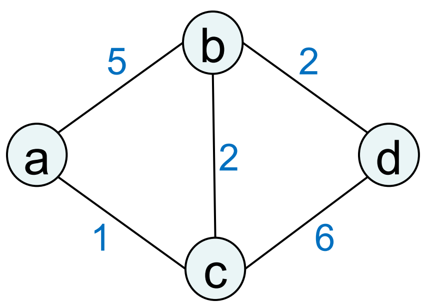
BFS treats all edges as having the same length.
This is rarely true in applications.

Denote the length of edge e = (u,v) by l(e) or l_e or l(u,v)

Extending BFS

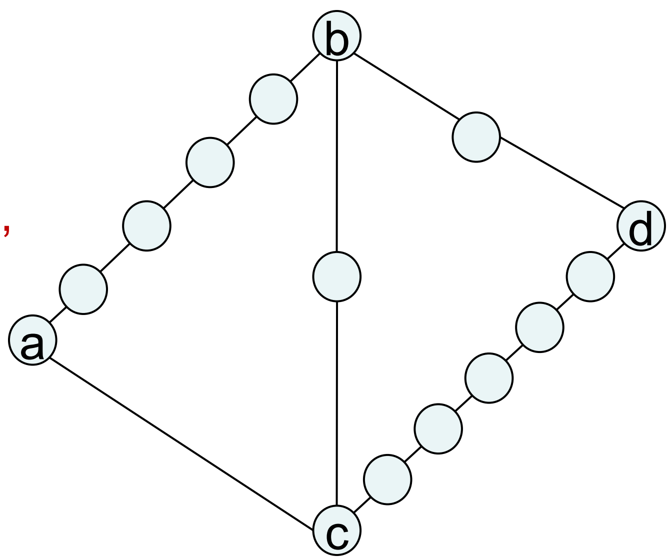
Suppose G has positive integral edge lengths

G



Simple trick: add *dummy nodes*

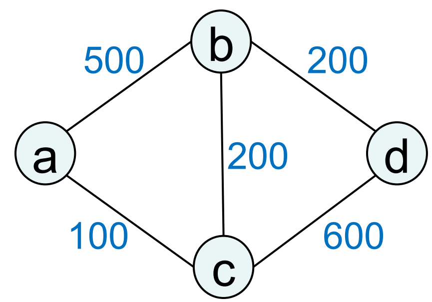
G'



(i) G' has unit-length edges
(ii) For the "real" nodes, distance in G = distance in G'

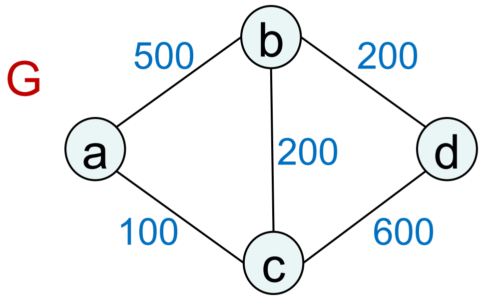
So run BFS on G' !

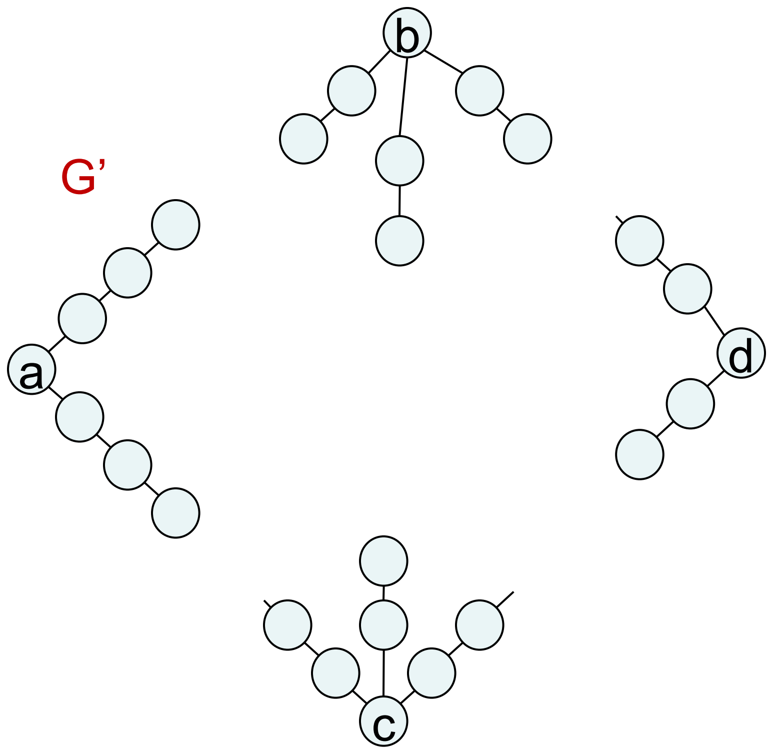
Problem: efficiency



If edge lengths in G are large:
(i) G' is enormous
(ii) BFS wastes a lot of time computing distances to dummy nodes we don't care about

Extending BFS





First 99 time steps: BFS (on G') slowly advances along a—b and a—c. Boring!

Can we snooze and have an alarm wake up us whenever BFS reaches a *real* node?

Alarm for each real node: estimated time of arrival based on edges currently being traversed.

T = 0	set alarms for b (500), c (100) snooze
T = 100	wake up, BFS is at c set alarms for b (300), d (700) snooze
T = 300	wake up, BFS is at b set alarm for d (500) snooze
T = 500	wake up, BFS is at d

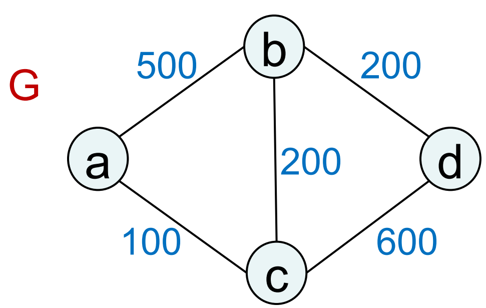
dist[c] = 100
dist[b] = 300
dist[d] = 500

Alarm clock algorithm

(Given graph G and starting node s)

set an alarm for node s at time 0
if the next alarm goes off at time T, for node u:
    distance[u] = T
    for each edge (u,v) in E:
        if no alarm for v, set one for T + l(u,v)
        if there is an alarm for v, but later than
        T + l(u,v), then reset to this earlier time

Exactly simulates BFS on G'...
we no longer need to construct G'!



How to implement alarm?
Answer: priority queue (aka heap)

A priority queue H stores:
- a set of elements (our nodes)
-associated key values (alarm times)
and supports these operations:

insert(H,x)	insert new element into H	set a new alarm
deletemin(H)	return element with smallest key value, remove from H	which alarm is going off next?
decreasekey(H,x)	allow x's key value to be decreased	allow alarm to be reset to an earlier time
makequeue(S)	make a queue out of the elements in S (and their keys)	initialize alarms

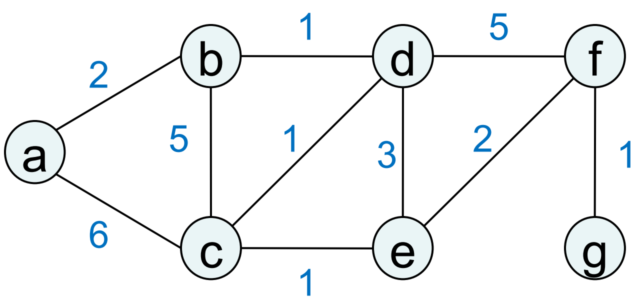
Dijkstra's algorithm

```
procedure dijkstra(G,l,s)

input: graph G = (V,E); node s;
  positive edge lengths le
output: for each node u, dist[u] is
  set to its distance from s

for u in V:
  dist[u] = ∞
dist[s] = 0
H = makequeue(V)  // key = dist[]

while H is not empty:
  u = deletemin(H)
  for each edge (u,v) in E:
    if dist[v] > dist[u] + l(u,v):
        dist[v] = dist[u] + l(u,v)
        decreasekey(H,v)
```

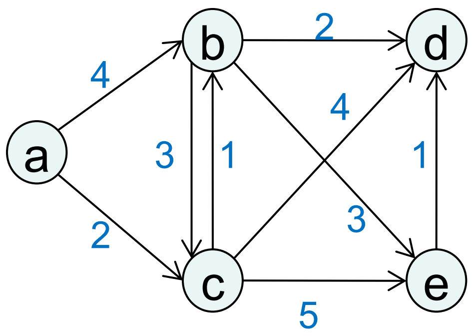


Another example

```
procedure dijkstra(G,l,s)

for u in V:
  dist[u] = ∞
  prev[u] = nil
dist[s] = 0
H = makequeue(V)  // key = dist[]

while H is not empty:
  u = deletemin(H)
  for each edge (u,v) in E:
    if dist[v] > dist[u] + l(u,v):
        dist[v] = dist[u] + l(u,v)
        prev[v] = u
        decreasekey(H,v)
```



Running time

```
procedure dijkstra(G,l,s)

for u in V:
  dist[u] = ∞
dist[s] = 0
H = makequeue(V)   // key = dist[]

while H is not empty:
  u = deletemin(H)
  for each edge (u,v) in E:
    if dist[v] > dist[u] + l(u,v):
        dist[v] = dist[u] + l(u,v)
        decreasekey(H,v)
```

Time:

O(V + E) +

V x deletemin +

V x insert +

E x decreasekey

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Running time

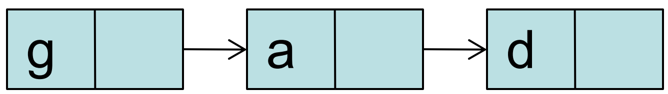
```
procedure dijkstra(G,l,s)

for u in V:
  dist[u] = ∞
dist[s] = 0
H = makequeue(V)   // key = dist[]

while H is not empty:
  u = deletemin(H)
  for each edge (u,v) in E:
    if dist[v] > dist[u] + l(u,v):
        dist[v] = dist[u] + l(u,v)
        decreasekey(H,v)
```

Time:
O(V + E) +
V x deletemin +
V x insert +
E x decreasekey

Depends on priority queue implementation:
eg. binary heap O(E log V)

Linked list implementation

Linked list, unordered



insert:

decreasekey:

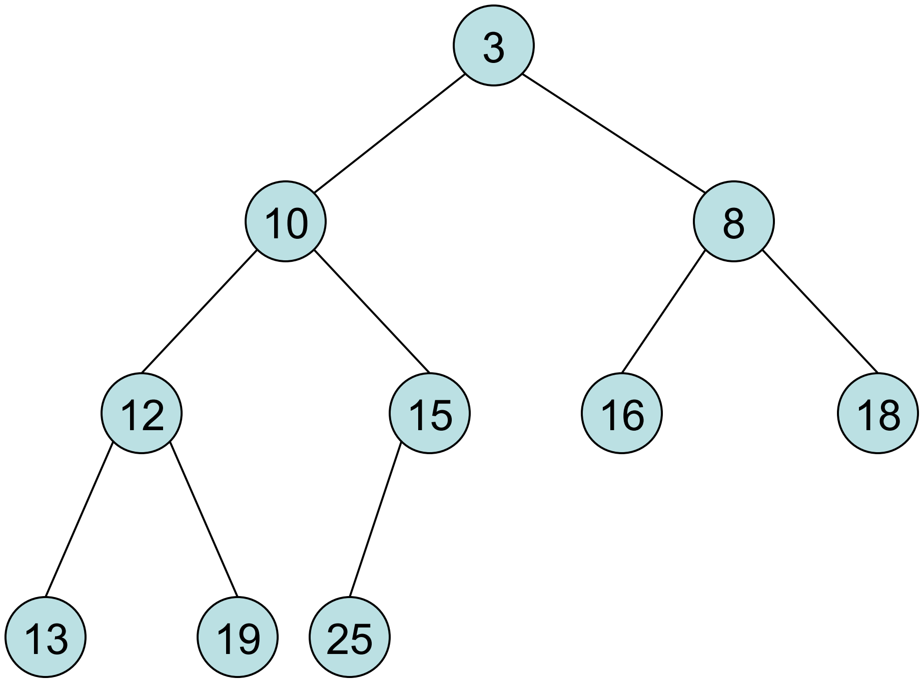
deletemin:

Binary heap

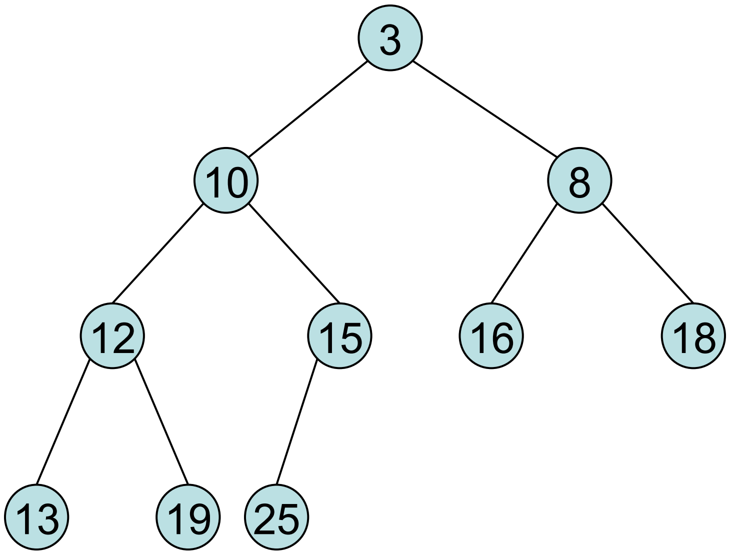
Complete binary tree: filled in row by row, left-to-right

Rule: each node's value is smaller than that of its children

Height $\leq \log_2 n + 1$



Binary heap



insert(7)

decreasekey(19 -> 6)

deletemin

d-ary heap

Same as a binary heap, but with
d children...

height:

insert

deletemin

Running time of Dijkstra's algorithm

	insert, decreasekey	**deletemin**	**V x deletemin + (V+E) x insert**
linked list	O(1)	O(V)	$O(V^2)$
binary heap	O(log V)	O(log V)	O((V+E) log V)
d-ary heap	$O(\log_d V)$	$O(d \log_d V)$	$O((dV + E) \log_d V)$
Fibonacci heap	O(1) amortized	O(log V)	O(E + V log V)

Which is best depends on *sparsity* of graph: ratio E/V (average degree).

Linked list vs. binary heap

Dense graph: E = £(V^2)

Linked list is better: $O(V^2)$

Sparse graph: E = O(V)

Binary heap is better: O(V log V)

d-ary heap

Best choice d $\approx$ E/V

Dense: $O(V^2)$

Sparse: O(V log V)

Intermediate: $E = V^{1+c}$

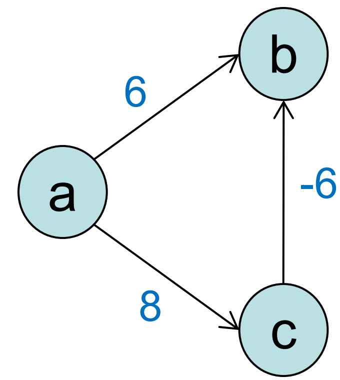
O(E/c), linear!

Dijkstra and negative edges

```
procedure dijkstra(G,l,s)

for u in V:
  dist[u] = ∞
dist[s] = 0
H = makequeue(V)   // key = dist[]

while H is not empty:
  u = deletemin(H)
  for each edge (u,v) in E:
    if dist[v] > dist[u] + l(u,v):
       dist[v] = dist[u] + l(u,v)
       decreasekey(H,v)
```

Basic principle of Dijkstra's algorithm: *the shortest path to any node only goes through nodes that are closer by*.

Not true if negative edges are present!



In Dijkstra's algorithm, dist[] values:
(i) are never too small
(ii) get changed only when updating along an edge:

```
procedure update(edge (u,v))
if dist[v] > dist[u] + l(u,v):
   dist[v] = dist[u] + l(u,v)
```