COL702: Advanced Data Structures and Algorithms

Ragesh Jaiswal, IITD

Introduction

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
  - Idea#1: Implement them on some platform, run and check.

Introduction

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
  - Idea#1: Implement them on some platform, run and check.
  - The speed of programs `P1` (implementation of `A1`) and `P2` (implementation of `A2`) may depend on various factors:
    - Input
    - Hardware platform
    - Software platform
    - Quality of the underlying algorithm

Introduction

- Idea#1: Implement them on some platform, run and check.
- Let `P1` denote implementation of `A1` and `P2` denote implementation of `A2`.
- Issues with Idea#1:
  - If `P1` and `P2` are run on different platforms; then the performance results are incomparable.
  - Even if `P1` and `P2` are run on the same platform, it does not tell us how `A1` and `A2` compare on some other platform.
  - There might be infinitely many inputs to compare the performance on.
  - There is the extra burden of implementing *both* algorithms, whereas what we wanted was first to figure out which one is better and then implement just that one.
- So, we need a platform-independent way of comparing algorithms.

Introduction

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- <u>Solution</u>:
  - Any algorithm is expressed in terms of basic operations such as *assignment, method call, arithmetic, comparison*.
  - For a fixed input, we will count the number of these basic operations in our algorithm. Suppose the number of these operations is b.
  - We will assume that the amount of time required to execute these basic operations is at most some constant T, which is independent of the input size.
  - The running time of the algorithm will be at most $(b \cdot T)$.

Introduction

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- Solution:
  - Any algorithm is expressed in terms of basic operations such as *assignment, method call, arithmetic, comparison*.
  - For a fixed input, we will count the number of these basic operations in our algorithm. Suppose the number of these operations is b.
  - We will assume that the amount of time required to execute these basic operations is at most some constant T, which is independent of the input size.
  - The running time of the algorithm will be at most $(b \cdot T)$.
  - **But, what about other inputs**? We are interested in measuring the performance of an algorithm and not the performance of an algorithm on a given input.

Introduction

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- Solution: Count the number of basic operations.
  - How do we measure performance for all inputs?

Example

`FindPositiveSum`(A, n)

- $sum \leftarrow 0$
- For $i = 1$ to n
  - if $(A[i] > 0)$ $sum \leftarrow sum + A[i]$
- return(sum)

- Note that the number of operations grows with the array size n.

Introduction

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- Solution: Count the number of basic operations.
  - How do we measure performance for all inputs?

Example

$\texttt{FindPositiveSum}(A, n)$

- $sum \leftarrow 0$
- For $i = 1$ to n
  - if $(A[i] > 0) sum \leftarrow sum + A[i]$
- return(sum)

- Note that the number of operations grows with the array size n.
- Even for all arrays of a fixed size n, the number of operations may vary depending on the numbers present in the array.

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- Solution: Count the number of basic operations.
  - How do we measure performance for all inputs?

Example

`FindPositiveSum`(A, n)
- $sum \leftarrow 0$
- For $i = 1$ to n
  - if $(A[i] > 0) sum \leftarrow sum + A[i]$
- return(sum)

- Note that the number of operations grows with the array size n.
- Even for all arrays of a fixed size n, the number of operations may vary depending on the numbers present in the array.
- For inputs of size n, we will count the number of operations in the worst-case. That is the number of operations for the worst-case input of size n.

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- Solution: Count the worst-case number of basic operations $b(n)$ for inputs of size n and then analyse how this *function* $b(n)$ behaves as n grows. This is known as worst-case analysis.

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- Solution: Count the worst-case number of basic operations $b(n)$ for inputs of size n and then analyse how this *function* $b(n)$ behaves as n grows. This is known as worst-case analysis.

Example

FindPositiveSum(A, n)	
- $sum \leftarrow 0$	[1 assignment]
- For $i = 1$ to n	[1 assignment + 1 comparison + 1 arithmetic]*n
- if $(A[i] > 0) sum \leftarrow sum + A[i]$	[1 assignment + 1 arithmetic + 1 comparison]*n
- return(sum)	[1 return]
	Total: $6n + 2$

Introduction

- Given two algorithms A1 and A2 for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- Solution: Count the worst-case number of basic operations $b(n)$ for inputs of size n and then analyse how this *function* $b(n)$ behaves as n grows. This is known as worst-case analysis.
- Few observations:
  - Usually, the running time grows with the input size n.
  - Consider two algorithm A1 and A2 for the same problem. A1 has a worst-case running time $(100n + 1)$ and A2 has a worst-case running time $(2n^2 + 3n + 1)$. Which one is better?
    - A2 runs faster for small inputs (e.g., $n = 1, 2$)
    - A1 runs faster for all large inputs (for all $n \geq 49$)

Introduction

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- Solution: Count the worst-case number of basic operations $b(n)$ for inputs of size n and then analyse how this *function* $b(n)$ behaves as n grows. This is known as worst-case analysis.
- Few observations:
  - Usually, the running time grows with the input size n.
  - Consider two algorithm `A1` and `A2` for the same problem. `A1` has a worst-case running time $(100n + 1)$ and `A2` has a worst-case running time $(2n^2 + 3n + 1)$. Which one is better?
    - `A2` runs faster for small inputs (e.g., $n = 1, 2$)
    - `A1` runs faster for all large inputs (for all $n \geq 49$)
  - We would like to make a statement independent of the input size. What is a meaningful solution?

Introduction

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- Solution: Count the worst-case number of basic operations $b(n)$ for inputs of size n and then analyse how this *function* $b(n)$ behaves as n grows. This is known as worst-case analysis.
- Observations regarding worst-case analysis:
  - Usually, the running time grows with the input size n.
  - Consider two algorithm `A1` and `A2` for the same problem. `A1` has a worst-case running time $(100n + 1)$ and `A2` has a worst-case running time $(2n^2 + 3n + 1)$. Which one is better?
    - `A2` runs faster for small inputs (e.g., $n = 1, 2$)
    - `A1` runs faster for all large inputs (for all $n \geq 49$)
  - We would like to make a statement independent of the input size.
  - Solution: Asymptotic analysis
    - We consider the running time for large inputs.
    - `A1` is considered better than `A2` since `A1` will beat `A2` **eventually**.

Introduction

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- Solution: Do an asymptotic worst-case analysis.

Introduction

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- Solution: Do an asymptotic worst-case analysis.
- Observations regarding asymptotic worst-case analysis:
  - It is difficult to count the number of operations at an extremely fine level.
  - Asymptotic analysis means that we are interested only in the **rate of growth** of the running time function w.r.t. the input size. For example, note that the rates of growth of functions $(n^2 + 5n + 1)$ and $(n^2 + 2n + 5)$ is determined by the n^2 (*quadratic*) term. The lower-order terms are insignificant. So, we may as well drop them.

Introduction

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- Solution: Do an asymptotic worst-case analysis.
- Observations regarding asymptotic worst-case analysis:
  - It is difficult to count the number of operations at an extremely fine level and keep track of these constants.
  - Asymptotic analysis means that we are interested only in the **rate of growth** of the running time function w.r.t. the input size. For example, note that the rates of growth of functions $(n^2 + 5n + 1)$ and $(n^2 + 2n + 5)$ is determined by the n^2 (*quadratic*) term. The lower-order terms are insignificant. So, we may as well drop them.
  - The nature of the growth rate of functions $2n^2$ and $5n^2$ are the same. Both are quadratic functions. It makes sense to drop these constants, too, when interested in the nature of the growth functions.
  - We need a notation to capture the above ideas.

Definition (Big-O)

Let $f(n)$ and $g(n)$ be functions mapping positive integers to positive real numbers. We say that $f(n)$ is $O(g(n))$ (or $f(n) = O(g(n))$ in short) **iff** there is a real constant $c > 0$ and an integer constant $n_0 \geq 1$ such that:

$$\forall n \geq n_0, f(n) \leq c \cdot g(n)$$

- Another short way of saying that $f(n) = O(g(n))$ is "$f(n)$ is **order of** $g(n)$".
- Show that: $8n + 5 = O(n)$.

Definition (Big-O)

Let $f(n)$ and $g(n)$ be functions mapping positive integers to positive real numbers. We say that $f(n)$ is $O(g(n))$ (or $f(n) = O(g(n))$ in short) **iff** there is a real constant $c > 0$ and an integer constant $n_0 \geq 1$ such that:

$$\forall n \geq n_0, f(n) \leq c \cdot g(n)$$

- Another short way of saying that $f(n) = O(g(n))$ is "$f(n)$ is **order of** $g(n)$".
- Show that: $8n + 5 = O(n)$.
  - For constants $c = 13$ and $n_0 = 1$,we show that $\forall n \geq n_0, 8n + 5 \leq 13 \cdot n$. So, by definition of big-O, $8n + 5 = O(n)$.

Definition (Big-O)

Let $f(n)$ and $g(n)$ be functions mapping positive integers to positive real numbers. We say that $f(n)$ is $O(g(n))$ (or $f(n) = O(g(n))$ in short) **iff** there is a real constant $c > 0$ and an integer constant $n_0 \geq 1$ such that:

$$\forall n \geq n_0, f(n) \leq c \cdot g(n)$$

- Another short way of saying that $f(n) = O(g(n))$ is "$f(n)$ is **order of** $g(n)$".
- $g(n)$ may be interpreted as an upper bound on $f(n)$.
- Show that: $8n + 5 = O(n)$.
- Is this true $8n + 5 = O(n^2)$?

Definition (Big-O)

Let $f(n)$ and $g(n)$ be functions mapping positive integers to positive real numbers. We say that $f(n)$ is $O(g(n))$ (or $f(n) = O(g(n))$ in short) **iff** there is a real constant $c > 0$ and an integer constant $n_0 \geq 1$ such that:

$$\forall n \geq n_0, f(n) \leq c \cdot g(n)$$

- Another short way of saying that $f(n) = O(g(n))$ is "$f(n)$ is **order of** $g(n)$".
- $g(n)$ may be interpreted as an upper bound on $f(n)$.
- <u>Show that</u>: $8n + 5 = O(n)$.
- Is this true $8n + 5 = O(n^2)$? Yes
- $g(n)$ may be interpreted as an *upper bound* on $f(n)$.
- How do we capture *lower bound*?

Definition (Big-Omega)

Let $f(n)$ and $g(n)$ be functions mapping positive integers to positive real numbers. We say that $f(n)$ is $\Omega(g(n))$ (or $f(n) = \Omega(g(n))$ in short) **iff** there is a real constant $c > 0$ and an integer constant $n_0 \geq 1$ such that:

$$\forall n \geq n_0, f(n) \geq c \cdot g(n)$$

Definition (Big-Omega)

Let $f(n)$ and $g(n)$ be functions mapping positive integers to positive real numbers. We say that $f(n)$ is $\Omega(g(n))$ (or $f(n) = \Omega(g(n))$ in short) **iff** there is a real constant $c > 0$ and an integer constant $n_0 \geq 1$ such that:

$$\forall n \geq n_0, f(n) \geq c \cdot g(n)$$

- Show that: $f(n) = \Omega(g(n))$ iff $g(n) = O(f(n))$.

Definition (Big-O)

Let $f(n)$ and $g(n)$ be functions mapping positive integers to positive real numbers. We say that $f(n)$ is $O(g(n))$ (or $f(n) = O(g(n))$ in short) **iff** there is a real constant $c > 0$ and an integer constant $n_0 \geq 1$ such that:

$$\forall n \geq n_0, f(n) \leq c \cdot g(n)$$

Definition (Big-Omega)

Let $f(n)$ and $g(n)$ be functions mapping positive integers to positive real numbers. We say that $f(n)$ is $\Omega(g(n))$ (or $f(n) = \Omega(g(n))$ in short) **iff** there is a real constant $c > 0$ and an integer constant $n_0 \geq 1$ such that:

$$\forall n \geq n_0, f(n) \geq c \cdot g(n)$$

- How do we say that $g(n)$ is both an upper bound and lower bound for a function $f(n)$? In other words, $g(n)$ is a **tight bound** on $f(n)$.

Definition (Big-O)

Let $f(n)$ and $g(n)$ be functions mapping positive integers to positive real numbers. We say that $f(n)$ is $O(g(n))$ (or $f(n) = O(g(n))$ in short) **iff** there is a real constant $c > 0$ and an integer constant $n_0 \geq 1$ such that:

$$\forall n \geq n_0, f(n) \leq c \cdot g(n)$$

Definition (Big-Omega)

Let $f(n)$ and $g(n)$ be functions mapping positive integers to positive real numbers. We say that $f(n)$ is $\Omega(g(n))$ (or $f(n) = \Omega(g(n))$ in short) **iff** there is a real constant $c > 0$ and an integer constant $n_0 \geq 1$ such that:

$$\forall n \geq n_0, f(n) \geq c \cdot g(n)$$

Definition (Big-Theta)

Let $f(n)$ and $g(n)$ be functions mapping positive integers to positive real numbers. We say that $f(n)$ is $\Theta(g(n))$ (or $f(n) = \Theta(g(n))$) **iff** $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

- Question: Show that $3n \log n + 4n + 5 \log n$ is $\Theta(n \log n)$.

- Growth rates:
  - Arrange the following functions in ascending order of growth rate:
    - n
    - $2^{\sqrt{\log n}}$
    - $n^{\log n}$
    - $2^{\log n}$
    - $n/\log n$
    - n^n

Introduction

- Given two algorithms `A1` and `A2` for a problem, how do we decide which one runs faster?
- What we need is a platform-independent way of comparing algorithms.
- Solution: Do an asymptotic worst-case analysis recording the running time using Big-(O, Ω, Θ) notation.

Introduction

- How do we describe an algorithm?
  - Using a pseudocode.
- What are the desirable features of an algorithm?
  1. It should be correct.
     - We use proof of correctness to argue correctness.
  2. It should run fast.
     - We do an asymptotic worst-case analysis noting the running time in Big-(O, Ω, Θ) notation and use it to compare algorithms.

End