1. There are n identical cookies to be divided between m cookie monsters. For every monster i, one knows how much happiness it will get on getting its j^{th} cookie. This information is given as $Happy[i, j]$, where $Happy$ is an $m \times n$ matrix. The happiness for any monster follows "*law of diminishing returns*". That is, for any monster i,
$$Happy[i, 1] \geq Happy[i, 2] \geq ... \geq Happy[i, n]$$
Design an algorithm that distributes the cookies to the monsters so that the total monster happiness is maximised. Give proof of correctness and running time analysis for your algorithm.

2. You are a chemical thief who raids a chemical shop that has n liquid chemicals that are mutually immiscible. You have a large vessel of total volume V that you want to use this to steal the chemicals. You know that the total volume of the i^{th} chemical available in the shop is $v(i)$ and the total value of this $v(i)$ volume of chemical is $w(i)$. You have to decide how much volume of each of the chemical to steal so as to maximize your profit. Design an algorithm that given inputs $V, v(1), w(1), v(2), w(2), ..., v(n), w(n)$ outputs the volume of items $(s_1, ..., s_n)$ to steal . Note that since you can only steal a total of V volume of liquids, $\sum_i s_i \leq V$.
   (a) Consider the greedy algorithm that considers chemicals in decreasing order of value $w(i)$'s and steals as much of an item as possible before considering the next item. Does this algorithm always give an optimal solution?
   (b) Consider the greedy algorithm that considers chemicals in increasing order of volume $v(i)$'s and steals as much of an item as possible before considering the next item. Does this algorithm always give an optimal solution?
   (c) Consider the greedy algorithm that considers chemicals in non-increasing order of *value per unit volume*. That is, the ratio $\frac{w(i)}{v(i)}$ and steals as much of an item as possible before considering the next item.
   We will show that this greedy algorithm always gives an optimal solution. For simplicity of argument, assume that $\frac{w(1)}{v(1)} \geq \frac{w(2)}{v(2)} \geq ... \geq \frac{w(n)}{v(n)}$. Let $(o_1, ..., o_n)$ be any optimal solution and let $(g_1, ..., g_n)$ be the output of our greedy algorithm. We will show that for all i:
$$g_1 \cdot \frac{w(1)}{v(1)} + ... + g_i \cdot \frac{w(i)}{v(i)} \geq o_1 \cdot \frac{w(1)}{v(1)} + ... + o_i \cdot \frac{w(i)}{v(i)} \qquad (1)$$
   This will prove that the greedy algorithm produces an optimal solution. To show this, first prove the following claim.
   Claim 1: For all i, $g_1 + g_2 + ... + g_i \geq o_1 + o_2 + ... + o_i$.

   Let us proceed assuming that the above claim holds. Let j be the item such that after stealing item j (as per the greedy algorithm), the vessel becomes full. Note that this means $g_{j+1} = 0, g_{j+2} = 0, ..., g_n = 0$. Let us do a case analysis based on the value of g_j. Consider the case when $g_j \leq o_j$. (*The case when $g_j > o_j$ is symmetric and is left as one of the exercises.*) The next claim shows inequality (1) when $i < j$. You have to prove this claim.

   Claim 2: The following holds for all $i < j$:
   (a) $g_1 \geq o_1, g_2 \geq o_2, ..., g_i \geq o_i$,
   (b) $g_1 \cdot \frac{w(1)}{v(1)} + ... + g_i \cdot \frac{w(i)}{v(i)} \geq o_1 \cdot \frac{w(1)}{v(1)} + ... + o_i \cdot \frac{w(i)}{v(i)}$.

Now, if $i \geq j$, then we can write:

$$
\begin{aligned}
(g_1 - o_1) + (g_2 - o_2) + ... + (g_{j-1} - o_{j-1}) &\geq (o_j - g_j) + (o_{j+1} - g_{j+1}) + ... + (o_i - g_i) \\
&\quad \text{(From Claim 1)} \\
\Rightarrow (g_1 - o_1) \cdot \frac{w(j)}{v(j)} + ... + (g_{j-1} - o_{j-1}) \cdot \frac{w(j)}{v(j)} &\geq (o_j - g_j) \cdot \frac{w(j)}{v(j)} + ... + (o_i - g_i) \cdot \frac{w(j)}{v(j)} \\
&\quad \text{(since all terms in the first inequality are } \geq 0) \\
\Rightarrow (g_1 - o_1) \cdot \frac{w(1)}{v(1)} + ... + (g_{j-1} - o_{j-1}) \cdot \frac{w(j-1)}{v(j-1)} &\geq (o_j - g_j) \cdot \frac{w(j)}{v(j)} + ... + (o_i - g_i) \cdot \frac{w(i)}{v(i)} \\
&\quad \text{(since } \frac{w(1)}{v(1)} \geq ... \geq \frac{w(i)}{v(i)}) \\
\Rightarrow g_1 \cdot \frac{w(1)}{v(1)} + ... + g_i \cdot \frac{w(i)}{v(i)} &\geq o_1 \cdot \frac{w(1)}{v(1)} + ... + o_i \cdot \frac{w(i)}{v(i)}
\end{aligned}
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