1. This is a recap. of a few proof techniques that you studied in the Discrete Mathematics course. We will use the following definition of even and odd numbers in the example problems that follow:

   Odd/even numbers: An integer n is called even iff there exists an integer k such that $n = 2k$. An integer n is called odd iff there exists an integer k such that $n = 2k + 1$.

   - Direct proof: Used for showing statements of the form p implies q. We assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true.
     - Give a direct proof of the statement: "If n is an odd, then n^2 is odd".
   - Proof by contraposition: Used for proving statements of the form p implies q. We take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\neg p$ must follow.
     - Prove by contraposition that "if n^2 is odd, then n is odd".
   - Proof by contradiction: Suppose we want to prove that a statement p is true and suppose we can find a contradiction q such that $\neg p$ implies q. Since q is false, but $\neg p$ implies q, we can conclude that $\neg p$ is false, which means that p is true. The contradiction q is usually of the form $r \wedge \neg r$ for some proposition r.
     - Give a proof by contradiction of the statement: "at least four of any 22 days must fall on the same day of the week"
   - Counterexample: Suppose we want to show that the statement for all x, $P(x)$ is false. Then we only need to find a counterexample, that is, an example x for which $P(x)$ is false.
     - Show that the statement "Every positive integer is the sum of squares of two integers" is false.
   - Mathematical Induction: This was discussed in the lecture.
     - Show using induction that for all positive integer n, $1 + 2 + 3 + ... + n = n \cdot (n + 1)/2$.
     - Show using induction that for all positive integers n, $1 + 2^1 + 2^2 + ... + 2^n = 2^{n+1} - 1$.

2. Solve the following recurrence relation and give the exact value of $T(n)$.

$$T(n) = \begin{cases} T(n-1) & \text{if } n > 1 \text{ and } n \text{ is odd} \\ 2 \cdot T(n/2) & \text{if } n > 1 \text{ and } n \text{ is even} \\ 1 & \text{if } n = 1 \end{cases}$$

$T(n) = 2^{\lfloor \log n \rfloor}$

Solution: The answer follows from the following claim.

Claim: For all $k \geq 0$, the following holds: For all $2^k \leq n < 2^{k+1}$, $T(n) = 2^k$.

Proof. We show this by induction on k. Let $P(k)$ denote the given proposition in the claim. We need to show that $\forall k, P(k)$ is true.

Base step: $P(0)$ holds since $T(1) = 1$.

Inductive step: Suppose $P(0), P(1), P(2), ..., P(i)$ are true. We will show that $P(i+1)$ is true. Consider any $2^{i+1} \leq n < 2^{i+2}$. We need to consider the case when n is even and n is odd.

If n is odd, then $T(n) = T(n-1) = 2 \cdot T(\frac{n-1}{2})$. Note that $2^{i+1} \leq n-1 < 2^{i+2}$. So, we have $2^i \leq (n-1)/2 < 2^{i+1}$. Applying induction hypothesis, we get $T(n) = 2^{i+1}$.

If n is even, then $T(n) = 2 \cdot T(n/2)$. Since $2^{i+1} \leq n < 2^{i+2}$, we have $2^i \leq n/2 < 2^{i+1}$. Applying induction hypothesis, we get that $T(n) = 2^{i+1}$. □

3. Consider functions $f(n) = 10n2^n + 3^n$ and $g(n) = n3^n$.

   Prove or disprove: $g(n)$ is $O(f(n))$

Solution: We will show that the statement is false.

Let us first recall the definition of big-O: For functions $h_1(n)$ and $h_2(n)$, $h_1(n)$ is $O(h_2(n))$ iff there exists constants $c > 0, n_0 > 0$ such that for all $n \geq n_0$ $h_1(n) \leq c \cdot h_2(n)$. So, $h_1(n)$ is not $O(h_2(n))$ iff for **all** constants $c > 0, n_0 > 0$, there exists $n \geq n_0$ such that $h_1(n) > c \cdot h_2(n)$.

Note that $(1/2)n3^n > c \cdot 3^n$ when $n > 2c$ for any constant $c > 0$. Moreover, note that $(1/2)n3^n > (c)10n2^n \Leftrightarrow (3/2)^n > 20c \Leftrightarrow n > \log_{3/2}(20c)$.

Combining the previous two statements, we get that for any constant $c > 0$, $n3^n > c \cdot (10n2^n + 3^n)$ when $n > \max\left(2c, \log_{3/2} 20c\right)$. This further implies that for any constants $c > 0, n_0 \geq 0$, $n'3^{n'} > c \cdot (10n'2^{n'} + 3^{n'})$ for $n' = \max\left(\lceil 2c \rceil + 1, \lceil \log_{3/2} 20c \rceil + 1, n_0\right)$. Note that $n' \geq n_0$. So, for any constants $c > 0, n_0 > 0$, there is a number $n \geq n_0$ (n' above is such a number) such that $n3^n > c \cdot (10n2^n + 3^n)$. This implies that $g(n)$ is not $O(f(n))$.