COL863: Quantum Computation and Information

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Quantum Computation: Phase estimation

Phase estimation

Suppose a unitary operator U has an eigenvector $|u\rangle$ with eigenvalue $e^{2\pi i\varphi}$. The goal is to estimate φ.

- We will use the assumption that there are black-boxes that:
  - prepare the state $|u\rangle$, and
  - perform the controlled-U^{2^j} operation.
- We will describe a phase estimation procedure that uses two registers:
  - A t-qubit register initially in state $|0...0\rangle$ (the value of t to be decided later), and
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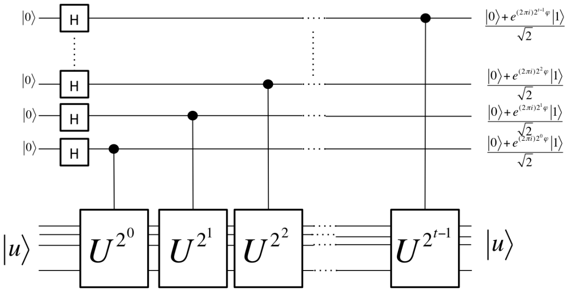

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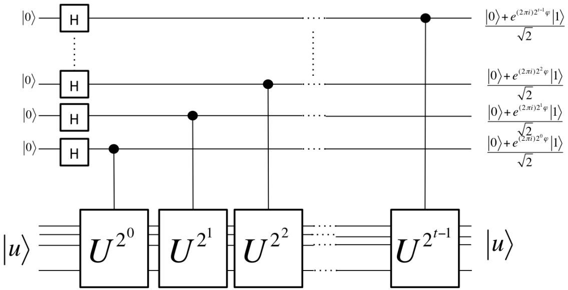

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- Let $0 \leq b \leq 2^t - 1$ be an integer such that $\frac{b}{2^t} = [0 \cdot b_1 \ldots b_t]$ is the best t bit approximation to φ that is less than φ. Let $\delta = \varphi - \frac{b}{2^t}$ (which implies $0 \leq \delta \leq 2^{-t}$).
- Claim 2.1: Applying the inverse Fourier transform on the first register in state $\frac{1}{2^{t/2}} \sum_{k=0}^{2^t - 1} e^{(2\pi i)\varphi k} |k\rangle$ ends in the following state:

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- Claim 2.2: Let α_l be the amplitude of $|(b + l) \ mod \ 2^t\rangle$. Then $\alpha_l = \frac{1}{2^t} \left(\frac{1 - e^{(2\pi i)(2^t \varphi - (b+l))}}{1 - e^{(2\pi i)(\varphi - (b+l)/2^t)}} \right) = \frac{1}{2^t} \left(\frac{1 - e^{(2\pi i)(2^t \delta - l)}}{1 - e^{(2\pi i)(\delta - l/2^t)}} \right)$.

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- The claim follows by setting $t = n + p$ and $\varepsilon = \frac{1}{2(2^p-1)}$. □

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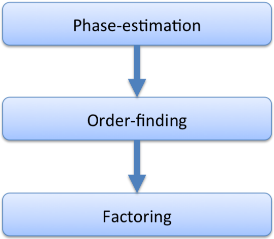
- The phase estimation protocol works when the second register is set to the eigenstate $|u\rangle$. In general, this may not be feasible.
- Observation: Any general state $|\psi\rangle$ may be written in terms of the eigenstates of U as $\sum_u c_u |u\rangle$.
- Exercise: The phase estimation procedure takes state $(|0\rangle)(\sum_u c_u |u\rangle)$ to $\sum_u c_u |\tilde{\varphi_u}\rangle |u\rangle$. If $t = n + \lceil \log\left(2 + \frac{1}{2\varepsilon}\right) \rceil$, then the probability of measuring φ_u accurate to n bits at the end of the phase estimation procedure is at least $|c_u|^2(1 - \varepsilon)$.

Phase estimation

Suppose a unitary operator U has an eigenvector $|u\rangle$ with eigenvalue $e^{2\pi i \varphi}$. The goal is to estimate φ.

- Phase estimation enables us to design quantum algorithms for the order-finding and factoring problems.



Quantum Computation: Order finding

- Given integers $N > x > 0$ such that x and N have no common factors, the order of x modulo N is defined to be the least positive integer r such that $x^r = 1 \ (mod \ N)$.
- Exercise: What is the order of 5 modulo 21?

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- Exercise: Is it an efficient algorithm?
- Let $L = \lceil \log n \rceil$. The number of bits needed to specify the problem is $O(L)$. So, an efficient algorithm should have running time that is polynomial in L.

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- Consider the operator U that has the following behaviour:

$$U|y\rangle \equiv \begin{cases} |xy \ (mod \ N)\rangle & \text{if } 0 \leq y \leq N-1 \\ |y\rangle & \text{if } N \leq y \leq 2^L - 1 \end{cases}$$

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- Exercise: Show that U is unitary.
- Exercise: Show that the states defined by

$$|u_s\rangle \equiv \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-(2\pi i)\frac{sk}{r}} \left| x^k \ (mod \ N) \right\rangle$$

are the eigenstates of U. Find the corresponding eigenvalues.

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- Exercise summary: Let $|u_s\rangle \equiv \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-(2\pi i)\frac{sk}{r}} \left|x^k \ (mod \ N)\right\rangle$ be an eigenstate of U. Then $U|u_s\rangle = e^{(2\pi i)\frac{s}{r}} |u_s\rangle$

End