COL863: Quantum Computation and Information

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Quantum Computation: Quantum Fourier transform

Discrete Fourier Transform (DFT)

The discrete Fourier transform takes as input a parameter N and a vector of complex numbers $x_0, ..., x_{N-1}$ and outputs a vector of complex numbers $y_0, ..., y_{N-1}$ where the inputs and outputs are related as:

$$y_k \equiv \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{\frac{2\pi i}{N} jk}$$

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- Question: Can we do this faster?

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- Question: Can we do this faster? Yes in $O(N \log N)$ operations using Fast Fourier Transform (FFT)
  - Claim 1: DFT can be computed by multiplying an $N \times N$ matrix W with the vector $X = (x_0, ..., x_{N-1})^T$, where $W_{ij} = w^{ij}$ and $w = e^{\frac{2\pi i}{N}}$.

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Let $X = (x_0, ..., x_{N-1})^T$ and W be an $N \times N$ matrix where $W_{ij} = w^{ij}$ and $w = e^{\frac{2\pi i}{N}}$. Then WX can be computed using $O(N \log N)$ operations.

Proof sketch

- The following picture captures the main idea of FFT.

$$WX = \begin{pmatrix} w^{(2j)k} & w^{(2j+1)k} \\ w^{(2j)k} & -w^{(2j+1)k} \end{pmatrix} \begin{pmatrix} X_{2j} \\ X_{2j+1} \end{pmatrix}$$

- The recurrence relation for the number of operations is given by $T(N) = 2T(N/2) + O(N)$ which gives $T(N) = O(N \log N)$.

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Quantum Fourier Transform (QFT)

The quantum Fourier transform on an orthonormal basis $|0\rangle, ..., |N-1\rangle$ is defined to be a linear operator with the following action on the basis states:

$$|j\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N} jk} |k\rangle .$$

Equivalently, the action on an arbitrary state is:

$$\sum_{j=0}^{N-1} x_j |j\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} y_k |k\rangle ,$$

where y_k is as in DFT.

- <u>Exercise</u>: Show that the Quantum Fourier transform operator is unitary.

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- Exercise: Show that the Quantum Fourier transform operator is unitary.
- Claim: Let $N = 2^n$. There is a quantum circuit of size $O(n^2)$ that computes the QFT on the computational basis corresponding to n-qubits.

QFT circuit

Let $N = 2^n$. There is a quantum circuit of size $O(n^2)$ that computes the QFT on the computational basis corresponding to n-qubits.

- For $j \in \{0, ..., N-1\}$, let $[j_1 j_2 ... j_n]$ be the binary representation of j. So, $j = j_1 2^{n-1} + j_2 2^{n-2} + ... + j_n 2^0$.
- We will also use binary fraction notation $[0 \cdot j_l ... j_m]$ which represents the number $\frac{j_l}{2} + \frac{j_{l+1}}{2^2} + \frac{j_m}{2^{m-l+1}}$.
- Claim 1: The QFT of a state $|j_1 ... j_n\rangle$ is given as below:

$$|j_1 ... j_n\rangle \rightarrow \left(\frac{|0\rangle + e^{2\pi i [0 \cdot j_n]}|1\rangle}{\sqrt{2}}\right) \left(\frac{|0\rangle + e^{2\pi i [0 \cdot j_{n-1} j_n]}|1\rangle}{\sqrt{2}}\right) ... \left(\frac{|0\rangle + e^{2\pi i [0 \cdot j_1 \cdots j_n]}|1\rangle}{\sqrt{2}}\right)$$

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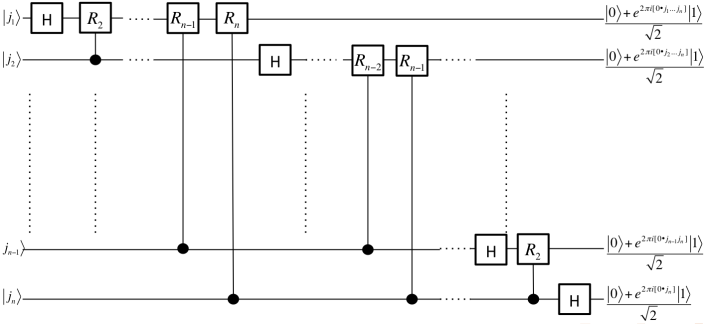
- This representation helps to construct the following circuit:



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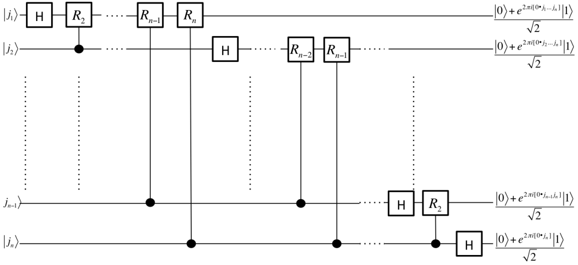

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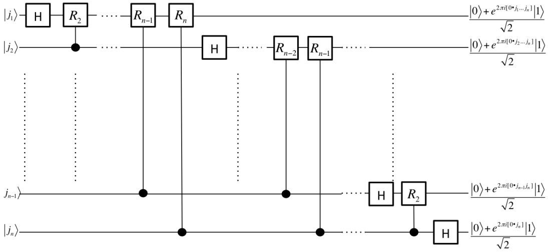

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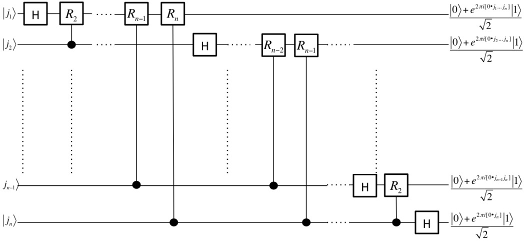

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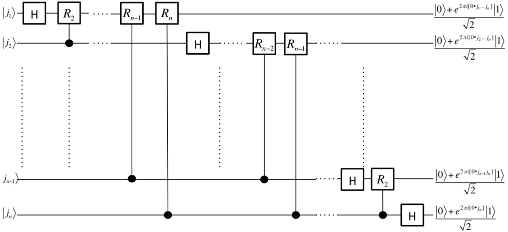

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- What about precision? Polynomial precision in each gate is sufficient

End