COL863: Quantum Computation and Information

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Quantum Computation: Quantum circuits

Quantum Circuit
Controlled operations

Theoerm

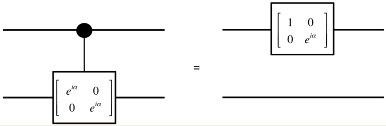
Suppose U is a unitary gate on a single qubit. Then there exist unitary operators A, B, C on a single qubit such that $ABC = I$ and $U = e^{i\alpha} AXBXC$, where α is some overall phase factor.

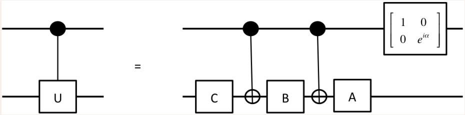
Question

For a single qubit U, can we implement Controlled-U gate using only CNOT and single-qubit gates? Yes

Construction sketch

The construction follows from the following circuit equivalences.





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For a single qubit U, can we implement Controlled-U gate with two control qubits using only `CNOT` and single-qubit gates?

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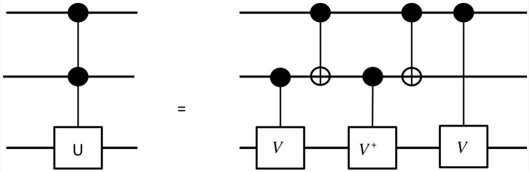

Here V is such that $V^2 = U$.

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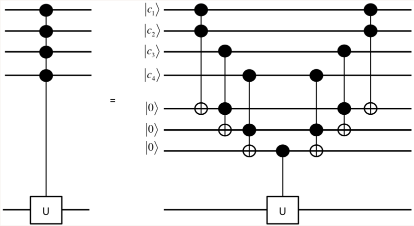
Question

For a single qubit U, can we implement Controlled-U gate with n control qubits using only CNOT and single-qubit gates? Yes using ancilla qubits

Construction sketch

An example construction with $n = 4$.



- A few other gates and circuit identities:

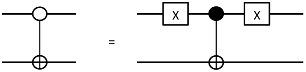
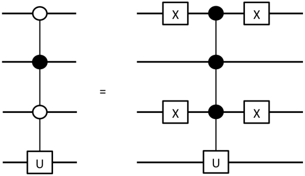
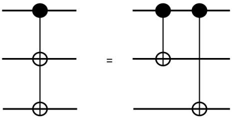


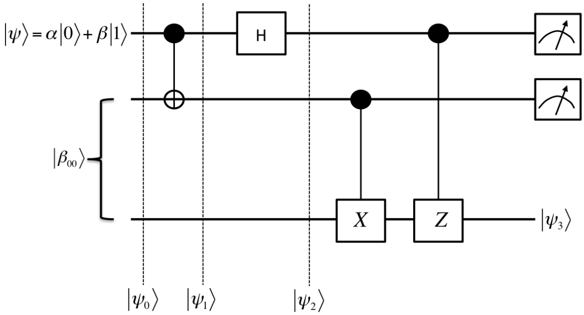
Figure: `NOT` gate applied to the target qubit conditional on the control qubit being 0.





Principle of deferred measurements

Measurements can always be moved from an intermediate stage of a quantum circuit to the end of the circuit; if the measurement results are used at any stage of the circuit, then the clasically controlled operations can be replaced by conditional quantum operations.



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Principle of implicit measurement

Without loss of generality, any unterminated quantum wires (qubits which are not measured) at the end of a quantum circuit may be assumed to be measured.

- A set of gates is said to be universal for quantum computation if **any** unitary operation may be **approximated** to arbitrary accuracy by a quantum circuit involving only those gates.

Claim

Any unitary operation can be approximated to arbitrary accuracy using Hadamard, CNOT, and $\pi/8$ gates.

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Proof sketch

- Claim 1: A single qubit operation may be **approximated** to arbitrary accuracy using the Hadamard, and $\pi/8$ gates.
- Claim 2: An arbitrary unitary operator may be expressed **exactly** using single qubit and CNOT gates.
  - Claim 2.1: An arbitrary unitary operator may be expressed **exactly** as a product of unitary operators that each acts non-trivially only on a subspace spanned by two computational basis states (such gates are called two-level gates).
  - Claim 2.2: An arbitrary two-level unitary operator may be expressed exactly using using single qubit and CNOT gates.

- What about efficiency?
  - Upper-bound: Any unitary can be approximated using exponentially many gates.
  - Lower-bound: There exists a unitary operation that which require exponentially many gates to approximate.

Claim 2.1

An arbitrary unitary operator may be expressed **exactly** as a product of unitary operators that each acts non-trivially only on a subspace spanned by two computational basis states.

Proof sketch

- The main idea can be understood using a 3×3 unitary matrix:

$$U = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & j \end{bmatrix}.$$

- We will find two-level unitary matrices U_1, U_2, U_3 such that

$$U_3 U_2 U_1 U = I \quad \text{and} \quad U = U_1^\dagger U_2^\dagger U_3^\dagger$$

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- Exercise
  - Show that any $d \times d$ unitary matrix can be written in terms of $d(d-1)/2$ two-level matrices.
  - There exists a $d \times d$ unitary matrix U which cannot be decomposed as a product of fewer than $d - 1$ two-level unitary matrices.

End