COL863: Quantum Computation and Information

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- Claim: (Projective measurement + unitary operators) = generalised measurement.

Proof sketch

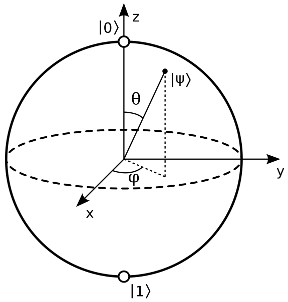
- Let Q be the state space of the quantum system in which we would like to make a generalised measurement using measurement operators M_m.
- We introduce an ancilla system with state space M with orthonormal basis $|m\rangle$.
- Let U be an operator defined as

$$U|\psi\rangle|0\rangle \equiv \sum_m M_m|\psi\rangle|m\rangle$$

  where $|0\rangle$ is an arbitrary state of M.
- Claim 1: U preserves inner products between states of the form $|\psi\rangle|0\rangle$.
- Claim 2: U can be extended to a unitary operator on $Q \otimes M$ (let us denote this by U itself).
- Claim 3: Let $P_m = I_Q \otimes |m\rangle\langle m|$. Projective measurement using P_m on $Q \otimes M$ is similar to generalised measurement using M_m on Q.

Quantum Computation: Quantum circuits

- Single qubit gates are 2×2 unitary matrices. Some of the important gates are:
  - Pauli matrices: $X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
  - Hadamard gate: $H \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.
  - Phase gate: $S \equiv \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$.
  - $\pi/8$ gate: $T \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$
- Simplification: A qubit $\alpha |0\rangle + \beta |1\rangle$ may be represented as $\cos\frac{\theta}{2}|0\rangle + e^{i\psi}\sin\frac{\theta}{2}|1\rangle$. So, any tuple (θ, ψ) represents a qubit.
- This has a nice visualisation in terms of Bloch sphere.



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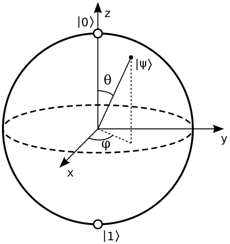

- The vector $(\cos\psi \sin\theta, \sin\psi \sin\theta, \cos\theta)$ is called the Bloch vector.

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- Pauli matrices give rise to three useful classes of unitary matrices when they are exponentiated, the rotational operators about the $\hat{x}$, $\hat{y}$, and $\hat{z}$ axis.

$$R_x(\theta) \equiv e^{-i\theta X/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} X = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_y(\theta) \equiv e^{-i\theta Y/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Y = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_z(\theta) \equiv e^{-i\theta Z/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

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- A few useful results:
  - Let $\hat{n} = (n_x, n_y, n_z)$ be a real unit vector. The rotation by θ about the $\hat{n}$ axis is given by

$$R_{\hat{n}}(\theta) \equiv e^{-i\frac{\theta}{2}(\hat{n}\cdot\vec{\sigma})} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2}(n_x X + n_y Y + n_z Z),$$

    where $\vec{\sigma}$ denotes the vector (X, Y, Z).
  - Theorem: Suppose U is a unitary operator on a single qubit. Then there exist real numbers α, β, γ, and δ such that $U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$.

Theorem

Suppose U is a unitary operator on a single qubit. Then there exist real numbers α, β, γ, and δ such that $U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$.

Proof sketch

There are real numbers $\alpha, \beta, \gamma, \delta$ such that:

$$U = \begin{bmatrix} e^{i(\alpha - \beta/2 - \delta/2)} \cos \frac{\gamma}{2} & -e^{i(\alpha - \beta/2 + \delta/2)} \sin \frac{\gamma}{2} \\ e^{i(\alpha + \beta/2 - \delta/2)} \sin \frac{\gamma}{2} & e^{i(\alpha + \beta/2 + \delta/2)} \cos \frac{\gamma}{2} \end{bmatrix}$$

Now one just needs to verify that the RHS matches $e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$.

Theorem

Suppose U is a unitary operator on a single qubit. Then there exist real numbers α, β, γ, and δ such that $U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$.

Theoerm

Suppose U is a unitary gate on a single qubit. Then there exist unitary operators A, B, C on a single qubit such that $ABC = I$ and $U = e^{i\alpha} AXBXC$, where α is some overall phase factor.

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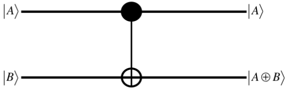
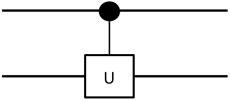
- Summary:
  - The above matrices are fundamental entities that define general classes of single-qubit unitary gates such that any single-qubit unitary gate can be represented in terms of these gates.

- The simplest two-qubit gate is the Controlled-NOT or `CNOT` gate:



  with matrix representation $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. The top qubit is called the control qubit and the bottom qubit is called the target qubit.
- Another simple two-qubit gate is the Controlled-U gate:



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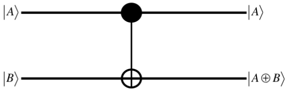

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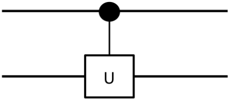

- <u>Some exercises</u>:
  - Build a `CNOT` gate from one Controlled-Z gate and two Hadamard gates.

Quantum Circuit
Controlled operations

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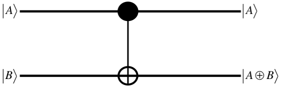

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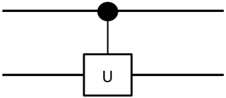

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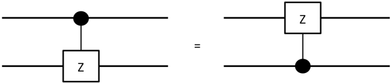
  - Show that:



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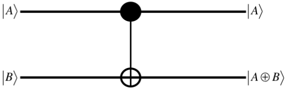

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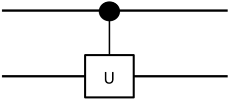

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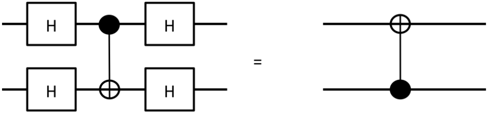

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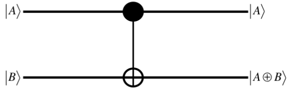

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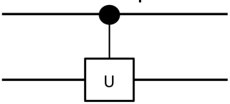

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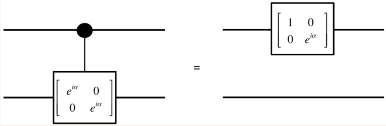
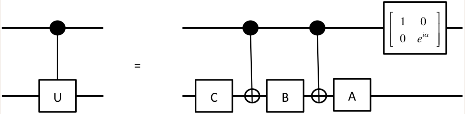
Construction sketch

The construction follows from the following circuit equivalences.





End