COL863: Quantum Computation and Information

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Distinguishing quantum states

Alice chooses a state $|\psi_i\rangle$ from a fixed set of states $|\psi_1\rangle, ..., |\psi_n\rangle$ (known to both Alice and Bob) and gives this state to Bob whose task is to identify i.

- Claim 2: There is no winning strategy for Bob if there are non-orthogonal states.

Proof sketch

- Assume $n = 2$ and let $|\psi_1\rangle$ and $|\psi_2\rangle$ be non-orthogonal.
- The most general strategy for Bob is to measure using operators $\{M_m\}$ and use a function $f : \{1, ..., m\} \rightarrow \{1, 2\}$ to return an answer to Alice. Suppose for the sake of contradiction, there exists such a winning strategy for Bob.
- Let $E_i = \sum_{j:f(j)=i} M_j^\dagger M_j$ for $i = 1, 2$.
- Since this is a winning strategy for Bob, we have: $\langle\psi_1| E_1 |\psi_1\rangle = 1; \langle\psi_2| E_2 |\psi_2\rangle = 1$
- Claim 2.1: $\sqrt{E_2}\,|\psi_1\rangle = 0$
- Claim 2.2: Decompose $|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\phi\rangle$, where $|\phi\rangle$ is orthonormal to $|\psi_1\rangle$. Then $|\beta| < 1$.
- Claim 2.3: $\langle\psi_2| E_2 |\psi_2\rangle = |\beta|^2 \langle\phi| E_2 |\phi\rangle \leq |\beta|^2 < 1$.
- The above contradicts with the fourth bullet item.

Superdense coding problem

Alice wants to send two classical bits to Bob. They share a Bell pair and the constraint is that Alice can only send a single qubit to Bob.

- Projective measurement is a special class of measurements and defines as special case of measurement postulate 3.
  - Is this a weaker notion than the generalized measurement postulate? No

Projective measurements

A projective measurement is described by an observable, M that is a Hermitian operator on the state space of the system being observed.

- The observable has a spectral decomposition $M = \sum_m mP_m$, where P_m is the projector onto the eigenspace of M with eigenvalue m.
- The possible outcomes of the measurement correspond to the eigenvalues, m, of the observable.
- The probability of measuring outcome being m on measuring state $|\psi\rangle$ is given by $p(m) = \langle\psi| P_m |\psi\rangle$.
- Given that m is the outcome of the measurement, the post-measurement state is $\frac{P_m|\psi\rangle}{\sqrt{p(m)}}$.

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- Observation: Generalized measurements where the measurement operators are constrained to be orthogonal projectors are the same as projective measurements.
- Exercise: M_m are orthogonal projectors if and only if M_m are Hermitian and $M_m M_{m'} = \delta_{m,m'} M_m$.

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- Exercise: M_m are orthogonal projectors if and only if M_m are Hermitian and $M_m M_{m'} = \delta_{m,m'} M_m$.
- Observation: Generalized measurements where the measurement operators M_m have additional constraints that M_m are Hermitian and $M_m M_{m'} = \delta_{m,m'} M_m$, are the same as projective measurements.

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- Claim: The average value of the measurement, denoted by $\mathbf{E}[M]$, is given by $\mathbf{E}[M] = \langle\psi| M |\psi\rangle$.

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- Exercise: Suppose we measure state ψ that is an eigenvector corresponding to eigenvalue m of the observable M. What is $\mathbf{E}[M]$?

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- Describing the observable M is one way to define the projective measurement. Other ways include:
  - A set of orthogonal projectors P_m satisfying completeness, that is, $\sum_m P_m = I$. The observable in this case is $\sum_m m P_m$.
  - An orthonormal basis $|m\rangle$ in which case, $P_m = |m\rangle \langle m|$.

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- Exercise: Discuss projective measurement of the state $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ w.r.t. observable Z.

- The measurement postulate defines rules for
  1. measurement statistics, and
  2. post-measurement state.
- For certain applications, the post-measurement state is not very important.
  - Can you think of such a scenario?
- POVM stands for Positive Operator-Valued Measure. The main ideas are captured in the following points:
  - For generalised measurement operators M_m and state $|\psi\rangle$, the measurement statistics are given by $p(m) = \langle\psi| M_m^\dagger M |\psi\rangle$.
  - Since we are interested only in the measurement statistics, it will be sufficient to describe the measurement using positive operators

$$E_m \equiv M_m^\dagger M_m$$

  - Observation: $\sum_m E_m = I$ and $p(m) = \langle\psi| E_m |\psi\rangle$.
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- Exercise: Let E_m be an arbitrary set of positive operators such that $\sum_m E_m = I$. Does there exist measurement operators M_m with the same measurement statistics are ones defined by E_m?
  - Yes. $M_m = \sqrt{E_m}$.

- POVM application: Show that the following POVM

$$
\begin{aligned}
E_1 &\equiv \frac{\sqrt{2}}{1+\sqrt{2}} \left|1\right\rangle \left\langle 1\right| \\
E_2 &\equiv \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(\left|0\right\rangle - \left|1\right\rangle)(\left\langle 0\right| - \left\langle 1\right|)}{2} \\
E_3 &\equiv I - E_1 - E_2
\end{aligned}
$$

helps to distinguish states $\left|0\right\rangle$ and $\frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}$ with the caveat that sometimes it may output "don't know".

Postulate 4

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_n\rangle$.

- We commented earlier that projective measurement is not a weaker notion when compared with generalised measurements (even though it may seem so).
- We will not argue that (Projective measurement + Unitary operators) has the same power generalised measurement.

Lemma

Suppose V is a Hilbert space with a subspace W. Suppose $U : W \rightarrow V$ is a linear operator that preserves inner products, that is, for any $|w_1\rangle, |w_2\rangle \in W$,

$$\langle w_1| U^\dagger U |w_2\rangle = \langle w_1|w_2\rangle .$$

Show that there exists a unitary operator $U' : V \rightarrow V$ that extends U. That is, $U' |w\rangle = U |w\rangle$ for all $|w\rangle \in W$ but U' is defined on the entire space V.

- Claim: (Projective measurement + unitary operators) = generalised measurement.

Proof sketch

- Let Q be the state space of the quantum system in which we would like to make a generalised measurement using measurement operators M_m.
- We introduce an ancilla system with state space M with orthonormal basis $|m\rangle$.
- Let U be an operator defined as

$$U|\psi\rangle|0\rangle \equiv \sum_m M_m|\psi\rangle|m\rangle$$

  where $|0\rangle$ is an arbitrary state of M.
- Claim 1: U preserves inner products between states of the form $|\psi\rangle|0\rangle$.

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- Claim 1: U preserves inner products between states of the form $|\psi\rangle|0\rangle$.
- Claim 2: U can be extended to a unitary operator on $Q \otimes M$ (let us denote this by U itself).
- Claim 3: Let $P_m = I_Q \otimes |m\rangle\langle m|$. Projective measurement using P_m on $Q \otimes M$ is similar to generalised measurement using M_m on Q.

End