COL351: Analysis and Design of Algorithms

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Course Overview

- Graph Algorithms
- Algorithm Design Techniques:
  - Greedy Algorithms
  - Divide and Conquer
  - Dynamic Programming
  - Network Flows
    - Hill-climbing and reduction

Network Flow

- Reduction
  1. We will obtain an algorithm A for a *Network Flow* problem using Hill-climbing.
  2. Given a new problem, we will *rephrase* this problem as a Network Flow problem.
  3. We will then use algorithm A to solve the rephrased problem and obtain a solution.
  4. Finally, we build a solution for the original problem using the solution to the rephrased problem.

- Hill-climbing optimization strategy:
  - Start with any solution that meets the constraints.
  - Repeat until there is no simple way to improve the solution:
    - Try to improve the solution via a "local" change, still satisfying the constraints.
  - Output the solution.
- A few points to note about Hill-climbing:
  - More often than not hill-climbing does NOT find an optimal solution, just a "local optimum"
  - Often used as an approximation algorithm or heuristic.
  - Also called *gradient ascent*, *interior point method*.

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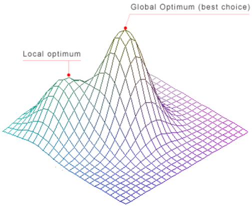
- Local optima:
  - One can view the set of all possible solutions as a high-dimensional region. The objective function then gives a height for each point.
  - We would like to find the highest point.
  - But we usually find a local optima, a point higher than others near it.
  - So, while global optima are local optima, the reverse is not always true.



- We want to model various kinds of networks using graphs and then solve real world problems with respect to these networks by studying the underlying graph.
- One problem that arises in network design is routing "flows" within the network.
  - Transportation Network: Vertices are cities and edges denote highways. Every highway has certain traffic capacity. We are interested in knowing the maximum amount commodity that can be shipped from a source city to a destination city.
  - Computer Networks: Edges are links and vertices are switches. Each link has some capacity of carrying packets. Again, we are interested in knowing how much traffic can a source node send to a destination node.

- To model these problems, we consider weighted, directed graph $G = (V, E)$ with the following properties:
  - Capacity: Associated with each edge e is a capacity that is a non-negative integer denoted by $c(e)$.
  - Source node: There is a source node s with no in-coming edges.
  - Sink node: There is a sink node t with no out-going edges. All other nodes are called *internal nodes*.

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    All other nodes are called *internal nodes*.
- Given such a graph, an "$s - t$ flow" in the graph is a function f that maps the edges to non-negative real numbers such that the following properties are satisfied:
  (a) Capacity constraint: For every edge e, $0 \leq f(e) \leq c(e)$.
  (b) Flow conservation: For every internal node v:

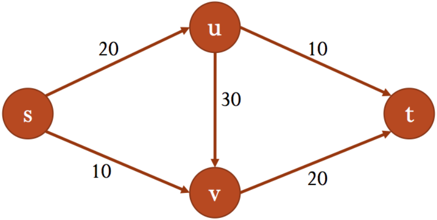
$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

Problem

Find an $s - t$ flow f in a given network graph such that the following quantity is maximized:

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

- Example:



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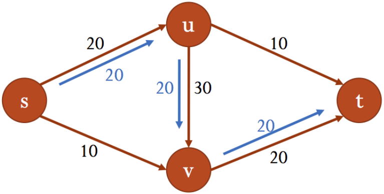

Figure: Routing 20 units of flow from s to t. Is it possible to "push more flow"?

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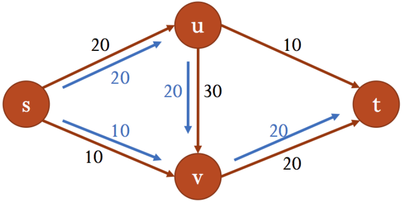

Figure: We should reset initial flow (u, v) to 10.

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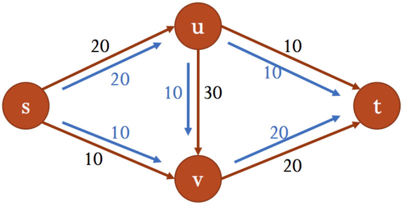
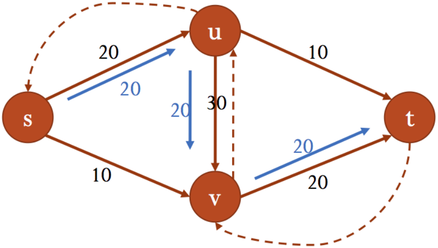


Figure: We should reset initial flow (u, v) to 10. Maximum flow from s is 30.

Approach

- We will iteratively build larger $s - t$ flows.
- Given an $s - t$ flow f, we will build a residual graph G_f that will allow us to reset flows along some of the edges.
- We will find an *augmenting path* in the residual graph G_f, push some flow along this path and update the flow f'.



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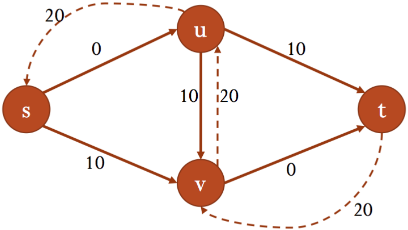

Figure: Graph G_f. $(f(s, u) = 20, f(s, v) = 0, f(u, v) = 20, f(u, t) = 0, f(v, t) = 20)$

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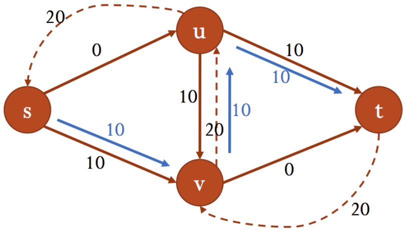

Figure: Augmenting path. $(f'(s, u) = 20, f'(s, v) = 10, f'(u, v) = 10, f'(u, t) = 10, f'(v, t) = 20)$

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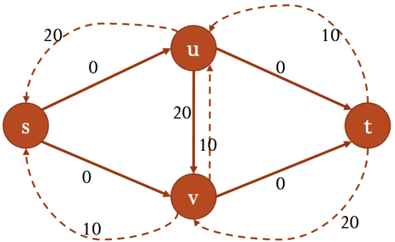


Figure: Graph $G_{f'}$. *(f'(s, u) = 20, f'(s, v) = 10, f'(u, v) = 10, f'(u, t) = 10, f'(v, t) = 20)*

- Residual graph G_f:
  - Forward edges: For every edge e in the original graph, there are $(c(e) - f(e))$ units of more flow we can send along that edge. So, we set the weight of this edge $(c(e) - f(e))$.
  - Backward edges: For every edge $e = (u, v)$ in the original graph, there are $f(e)$ units of flow that we can undo. So we add a reverse edge $e' = (v, u)$ and set the weight of e' to $f(e)$.

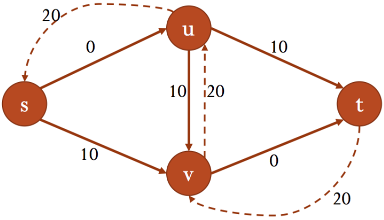
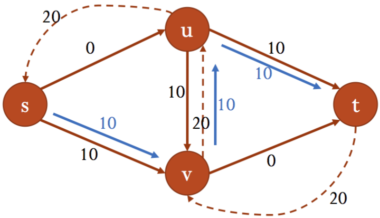


Figure: Graph G_f. *(f(s, u) = 20, f(s, v) = 0, f(u, v) = 20, f(u, t) = 0, f(v, t) = 20)*

- Augmenting flow in G_f:
  - Let P be a simple $s - t$ path in G_f. Note that this contains forward and backward edges.
  - Let e_{min} be an edge in the path P with minimum weight w_{min}
  - For every forward edge e in P, set $f'(e) \leftarrow f(e) + w_{min}$
  - For every backward edge (x, y) in P, set $f'(y, x) \leftarrow f(y, x) - w_{min}$
  - For all remaining edges e, $f'(e) = f(e)$



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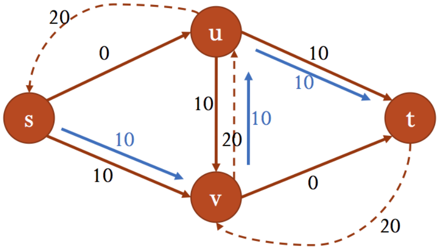
- Claim 1: f' is an $s - t$ flow.
- Proof sketch:
  - Capacity constraint for each edge is satisfied.
  - Flow conservation at each vertex is satisfied.



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Algorithm

`Ford-Fulkerson`

- Start with a flow f such that $f(e) = 0$
- While there is an $s - t$ path P in G_f
  - Augment flow along an $s - t$ path and let f' be resulting flow
  - Update f to f' and G_f to $G_{f'}$
- return(f)

- What is the running time of the above algorithm?

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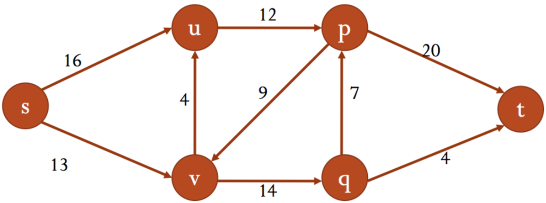

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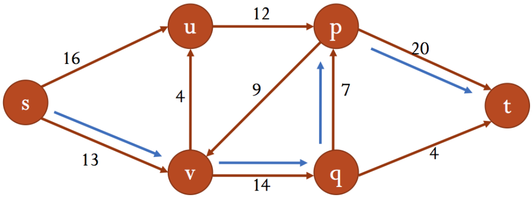

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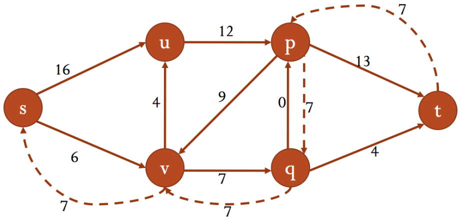

Figure: Graph G_f, where $f(s,u) = 0, f(s,v) = 7, f(v,u) = 0, f(v,q) = 7, f(u,p) = 0, f(p,v) = 0, f(p,t) = 7, f(q,p) = 7, f(q,t) = 0$

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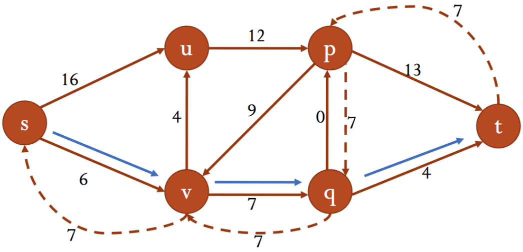

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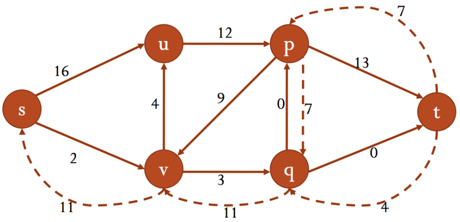

Figure: Graph G_f, where $f(s,u) = 0, f(s,v) = 11, f(v,u) = 0, f(v,q) = 11, f(u,p) = 0, f(p,v) = 0, f(p,t) = 7, f(q,p) = 7, f(q,t) = 4$

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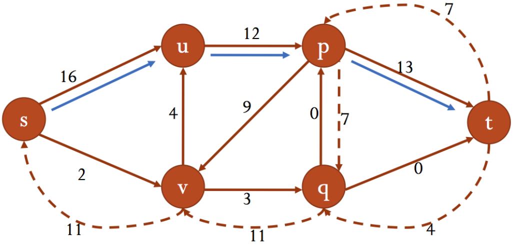

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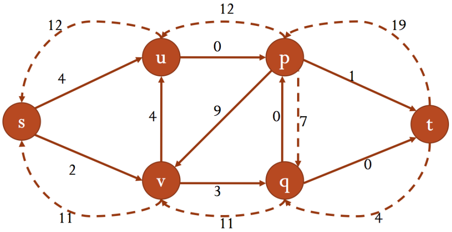

Figure: Graph G_f, where $f(s,u) = 12, f(s,v) = 11, f(v,u) = 0, f(v,q) = 11, f(u,p) = 12, f(p,v) = 0, f(p,t) = 19, f(q,p) = 7, f(q,t) = 4$

End