COL863: Quantum Computation and Information

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Quantum Computation: Phase estimation

Phase estimation

Suppose a unitary operator U has an eigenvector $|u\rangle$ with eigenvalue $e^{2\pi i\varphi}$. The goal is to estimate φ.

- We will use the assumption that there are black-boxes that:
  - prepare the state $|u\rangle$, and
  - perform the controlled-U^{2^j} operation.
- We will describe a phase estimation procedure that uses two registers:
  - A t-qubit register initially in state $|0...0\rangle$ (the value of t to be decided later), and
  - a register that begins in the state $|u\rangle$.

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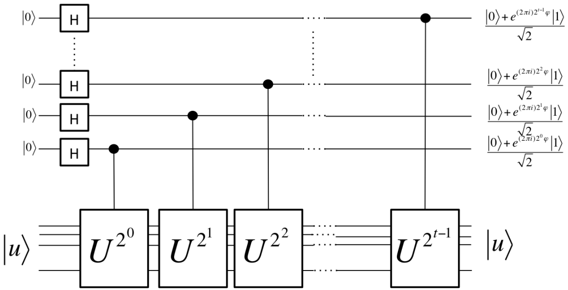

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- Question: Suppose φ may be expressed exactly as $\varphi = [0 \cdot \varphi_1\varphi_2 \ldots \varphi_t]$. Suggest a way to retrieve the value of φ?
    - Taking the inverse-fourier transform and measuring the value of the first register in the computational basis gives φ.
- In general, we will show that the inverse Fourier transform has the following behaviour:

$$\frac{1}{2^{t/2}}\sum_{j=0}^{2^t-1} e^{(2\pi i)\varphi j}|j\rangle|u\rangle \rightarrow |\tilde{\varphi}\rangle|u\rangle$$

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Proof sketch

- Let $0 \leq b \leq 2^t - 1$ be an integer such that $\frac{b}{2^t} = [0 \cdot b_1 \ldots b_t]$ is the best t bit approximation to φ that is less than φ. Let $\delta = \varphi - \frac{b}{2^t}$ (which implies $0 \leq \delta \leq 2^{-t}$).
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- Claim 2.2: Let α_l be the amplitude of $|(b + l) \ mod \ 2^t\rangle$. Then $\alpha_l = \frac{1}{2^t} \left(\frac{1 - e^{(2\pi i)(2^t \varphi - (b+l))}}{1 - e^{(2\pi i)(\varphi - (b+l)/2^t)}} \right) = \frac{1}{2^t} \left(\frac{1 - e^{(2\pi i)(2^t \delta - l)}}{1 - e^{(2\pi i)(\delta - l/2^t)}} \right)$.

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- The claim follows by setting $t = n + p$ and $\varepsilon = \frac{1}{2(2^p-1)}$. □

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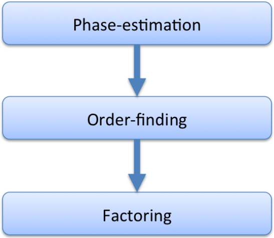
Suppose a unitary operator U has an eigenvector $|u\rangle$ with eigenvalue $e^{2\pi i\varphi}$. The goal is to estimate φ.

- The phase estimation protocol works when the second register is set to the eigenstate $|u\rangle$. In general, this may not be feasible.
- Observation: Any general state $|\psi\rangle$ may be written in terms of the eigenstates of U as $\sum_u c_u |u\rangle$.
- Exercise: The phase estimation procedure takes state $(|0\rangle)(\sum_u c_u |u\rangle)$ to $\sum_u c_u |\tilde{\varphi_u}\rangle |u\rangle$. If $t = n + \lceil \log\left(2 + \frac{1}{2\varepsilon}\right)\rceil$, then the probability of measuring φ_u accurate to n bits at the end of the phase estimation procedure is at least $|c_u|^2(1-\varepsilon)$.

Phase estimation

Suppose a unitary operator U has an eigenvector $|u\rangle$ with eigenvalue $e^{2\pi i\varphi}$. The goal is to estimate φ.

- Phase estimation enables us to design quantum algorithms for the order-finding and factoring problems.



Quantum Computation: Order finding

- Given integers $N > x > 0$ such that x and N have no common factors, the order of x modulo N is defined to be the least positive integer r such that $x^r = 1 \ (mod \ N)$.
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Given co-prime integers $N > x > 0$, compute the order of x modulo N.

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- Exercise: Is it an efficient algorithm?
- Let $L = \lceil \log n \rceil$. The number of bits needed to specify the problem is $O(L)$. So, an efficient algorithm should have running time that is polynomial in L.

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Given co-prime integers $N > x > 0$, compute the order of x modulo N.

- Consider the operator U that has the following behaviour:

$$U|y\rangle \equiv \begin{cases} |xy \ (mod \ N)\rangle & \text{if } 0 \leq y \leq N-1 \\ |y\rangle & \text{if } N \leq y \leq 2^L - 1 \end{cases}$$

- Exercise: Show that U is unitary.

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- Exercise: Show that U is unitary.
- Exercise: Show that the states defined by

$$|u_s\rangle \equiv \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-(2\pi i)\frac{sk}{r}} \left| x^k \pmod N \right\rangle$$

are the eigenstates of U. Find the corresponding eigenvalues.

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Modular exponentiation

Given $|z\rangle |y\rangle$, design a circuit that ends in the state $|z\rangle |x^z y \ (mod \ N)\rangle$.

- What we wanted to do was $|z\rangle |y\rangle \rightarrow |z\rangle U^{z_t 2^{t-1}} ... U^{z_1 2^0} |y\rangle$ but then this is the same as $|z\rangle |x^z y \ (mod \ N)\rangle$.
- Question: Suppose we work with the first register being of size $t = 2L + 1 + \lceil \log{(2 + \frac{1}{2\varepsilon})} \rceil = O(L)$. What would be the size of the circuit?

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  - How do we implement controlled U^{2^j}? Modular exponentiation
  - How do we prepare an eigenstate $|u_s\rangle$?
    - We work with $|1\rangle$ as the first register since $\frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} |u_s\rangle = |1\rangle$.

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  - Question: How do we extract r from this? Continued fractions

Continued fraction

A finite simple continued fraction is defined by a collection of positive integers $a_0, ..., a_N$:

$$[a_0, ..., a_N] \equiv a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_N}}}}$$

The n^{th} convergent ($0 \leq n \leq N$) of this continued fraction is defined to be $[a_0, ..., a_n]$.

- Theorem: Suppose $x \geq 1$ is a rational number. Then x has a representation as a continued fraction, $x = [a_0, ..., a_N]$. This may be found by the continued fraction algorithm.
- Exercise: Find the continued fraction expansion of $\frac{31}{13}$.
- Question: What is the running time for the continued fractions algorithm for any given rational number $\frac{p}{q} \geq 1$?

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- Question: What is the running time for the continued fractions algorithm for any given rational number $\frac{p}{q} \geq 1$?
- Theorem: Let $a_0, ..., a_N$ be a sequence of positive numbers. Then $[a_0, ..., a_n] = \frac{p_n}{q_n}$, where p_n and q_n are real numbers defined inductively by $p_0 \equiv 0$, $q_0 \equiv 1$, $p_1 \equiv 1 + a_0 a_1$, $q_1 \equiv a_1$, and for $2 \leq n \leq N$,

$$\begin{aligned} p_n &\equiv a_n p_{n-1} + p_{n-2} \\ q_n &\equiv a_n q_{n-1} + q_{n-2} \end{aligned}$$

  In the case when a_j are positive integers, so too are p_j and q_j and moreover $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$ for $n \geq 1$ which implies that $gcd(p_n, q_n) = 1$.

End