COL863: Quantum Computation and Information

Ragesh Jaiswal, CSE, IIT Delhi

Quantum Mechanics: Density operator

- The trace of a square matrix is defined to be the sum of diagonal elements. That is:

$$tr(A) \equiv \sum_i A_{ii}$$

  - Exercise: Show that $tr(AB) = tr(BA)$.
  - Exercise: Show that $tr(A + B) = tr(A) + tr(B)$.
  - Exercise: Show that $tr(zA) = ztr(A)$.
  - Exercise: Show that the trace operator is invariant under change of basis.
  - Exercise: Show that for any orthonormal basis $|i\rangle$,

$$tr(A) = \sum_i \langle i| A |i\rangle .$$

  - Exercise: Show that for any unit vector $|\psi\rangle$,

$$tr(A |\psi\rangle \langle\psi|) = \langle\psi| A |\psi\rangle .$$

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- An alternative and mathematically equivalent formulation is through density operators and density matrices.
    - Why: Talking about individual subsystems of a composite system becomes simpler.

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    - Why: Talking about individual subsystems of a composite system becomes simpler.
- The density operator is used to describe an ensemble of pure states $\{p_i, |\psi_i\rangle\}$. That is, a quantum system that is in state $|\psi_i\rangle$ with probability p_i.
    - Question Can you give a scenario where it may be useful to describe such an ensemble of states?
- Density operator: The density operator of such a system is defined by:

$$\rho \equiv \sum_i p_i |\psi_i\rangle \langle\psi_i|$$

- The density operator is often known as density matrix.

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The density operator for an ensemble of pure states $\{p_i, |\psi_i\rangle\}$ is given by $\rho \equiv \sum_i p_i |\psi_i\rangle \langle\psi_i|$.

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- Claim 1: Under a unitary operator U, the density operator evolves as: $\rho \overset{U}{\rightarrow} U\rho U^\dagger$.
- Claim 2: Making a generalised measurement using measurement operators M_m satisfies the following measurement statistics:

$$p(m|i) = tr(M_m^\dagger M_m |\psi_i\rangle \langle\psi_i|); \quad p(m) = tr(M_m^\dagger M_m \rho).$$

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  where $p(m|i)$ denotes the conditional probability of measurement output being m given that the state was $|\psi_i\rangle$.
- Claim 3: Suppose m was the measurement output, then the post measurement density operator is $\rho_m = \frac{M_m \rho M_m^\dagger}{tr(M_m^\dagger M_m \rho)}$.

- What are the necessary and sufficient conditions for an operator ρ to be a density operator w.r.t. some ensemble $\{p_i, |\psi_i\rangle\}$?

Theorem (Characterization of density operators)

An operator ρ is the density operator associated to some ensemble $\{p_i, |\psi_i\rangle\}$ if and only if it satisfies the conditions:

1. (Trace condition) $tr(\rho) = 1$.
2. (Positivity condition) ρ is a positive operator.

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Postulate 1

Associated to any isolated physical system is a complex vector space with inner products known as the state space of the system. The system is completely described by its density operator which is a positive operator ρ with trace 1. If a quantum system is in state ρ_i with probability p_i, then the density operator of the system is $\sum_i p_i \rho_i$.

Postulate 2

The evolution of a closed quantum system if described by a unitary transformation. That is, the state ρ of the system at time t_1 is related to the state ρ' of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2, $\rho' = U \rho U^\dagger$.

Postulate 3

Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. These are operator acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is ρ immediately before the measurement, then the probability that result m occurs is given by $p(m) = tr(M_m^\dagger M_m \rho)$, and the state of the system after measurement is $\frac{M_m \rho M_m^\dagger}{tr(M_m^\dagger M_m \rho)}$. The measurement operators satisfy the completeness equation, $\sum_m M_m^\dagger M_m = I$.

Postulate 4

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system i is prepared in state ρ_i, then the joint state of the total system is $\rho_1 \otimes \rho_2 \otimes .... \otimes \rho_n$.

- Pure and mixed state: A quantum system whose state $|\psi\rangle$ is known exactly is said to be in a pure state. In this case, the density operator is simply $\rho = |\psi\rangle\langle\psi|$. Otherwise ρ is in a mixed state.
- Exercise: $tr(\rho^2) \leq 1$.

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  - Consider the following density matrix $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$.
  - Following are two different ensembles:
    1. $\{(3/4, |0\rangle), (1/4, |1\rangle)\}$
    2. $\{(1/2, |a\rangle), (1/2, |b\rangle)\}$, where $|a\rangle = \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle$ and $|b\rangle = \sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle$.

- Question: Is there a unique ensemble of quantum states represented by any density matrix ρ? No
- Question: Is there a characterisation of the *class* of ensembles that generate a particular density matrix?

Theorem

The sets $\left|\tilde{\psi}_i\right\rangle$ *and* $\left|\tilde{\phi}_i\right\rangle$ *generate the same density matrix if and only if:*

$$\left|\tilde{\psi}_i\right\rangle = \sum_j u_{ij} \left|\tilde{\phi}_j\right\rangle$$

where u_{ij} *is a unitary matrix of complex numbers, with indices* i *and* j*, and we pad whichever set of vectors is smaller with additional vectors* $\mathbf{0}$ *so that the two sets have the same number of elements.*

- Corollary: For normalized sates $|\psi_i\rangle$ and $|\phi_j\rangle$ with probability distributions p_i and q_j, we have $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| = \sum_j q_j |\phi_j\rangle \langle\phi_j|$ if and only if $\sqrt{p_i}\, |\psi_i\rangle = \sum_j u_{ij} \sqrt{q_j}\, |\phi_j\rangle$.

- Suppose we have physical systems A and B whose joint state is described by a density operator ρ^{AB}. The reduced density operator for system A is defined by

$$\rho^A \equiv tr_B(\rho^{AB}).$$

  - tr_B is a map of operators known as the partial trace over system B. The partial trace is defined by

$$tr_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2| \, tr(|b_1\rangle\langle b_2|) = (\langle b_2|\,|b_1\rangle)\,|a_1\rangle\langle a_2|\,.$$

    Furthermore, partial trace is linear in its inputs.

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- Exercise: Let A and B be single qubit systems which is in the joint state $|01\rangle$. What is the density operator ρ? What is the reduced density operator ρ^A?

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    Furthermore, partial trace is linear in its inputs.
- How do we interpret the reduced density operator? What significance does it have?

Significance of partial trace

The partial trace is the unique operation which gives rise to the correct description of observable quantities for subsystems of a composite system.

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- Example 1: Let ρ, σ be density operators for systems A, B respectively. Then

$$tr_B(\rho \otimes \sigma) = \rho tr(\sigma) = \rho.$$

- Example 2: Let a two qubit system be in the Bell state $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$. What is the reduced density operator of the first qubit?

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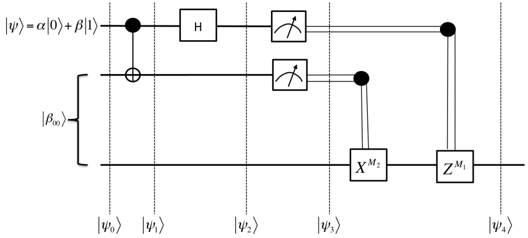
- Example 3: Consider Quantum teleportation.



  - What is the density operator just before Alice makes the measurements?
  - What is the reduced density operator for Bob's system?

End