COL863: Quantum Computation and Information

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Quantum Mechanics: Postulates

- The postulates of quantum mechanics were derived after a long process of trial and error.

Postulate 1 (State space)

Associated to any isolated physical system is a complex vector space with inner product (Hilbert space) known as the *state space* of the system. The system is completely described by its *state vector*, which is a unit vector in the system's state space.

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- We start with a simplest quantum mechanical system (a qubit) that has a two-dimensional state space with $|0\rangle$ and $|1\rangle$ being the orthonormal basis. This system is described by a state vector $|\psi\rangle$ where $\langle\psi|\psi\rangle = 1$.

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Postulate 2 (Evolution)

The evolution of a *closed* quantum system is described by a *unitary transformation*. That is, the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at time t_2 by a unitary operator U which only depends on the times t_1 and t_2, $|\psi'\rangle = U|\psi\rangle$.

- Doesn't applying a unitary gate contradict with the system being closed?

Postulate 3 (Measurement)

Quantum measurements are described by a collection $\{M_m\}$ of *measurement operators*. These are operators acting on the state space of the system being measured. The following properties hold:

- The index m refers to the measurement outcomes that may occur in the experiment.
- If the state of the system is $|\psi\rangle$ immediately before the measurement, then the probability that the result m occurs is given by
$$p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle ,$$
and the state of the system after the measurement is given by
$$\frac{M_m |\psi\rangle}{\sqrt{\langle\psi| M_m^\dagger M_m |\psi\rangle}}$$
- The measurement operators satisfy the *completeness equation*,
$$\sum_m M_m^\dagger M_m = I$$
.

- Projective measurement is a special class of measurements and defines as special case of measurement postulate 3.
  - Is this a weaker notion than the generalized measurement postulate? No

Projective measurements

A projective measurement is described by an observable, M that is a Hermitian operator on the state space of the system being observed.

- The observable has a spectral decomposition $M = \sum_m mP_m$, where P_m is the projector onto the eigenspace of M with eigenvalue m.
- The possible outcomes of the measurement correspond to the eigenvalues, m, of the observable.
- The probability of measuring outcome being m on measuring state $|\psi\rangle$ is given by $p(m) = \langle\psi| P_m |\psi\rangle$.
- Given that m is the outcome of the measurement, the post-measurement state is $\frac{P_m|\psi\rangle}{\sqrt{p(m)}}$.

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- Observation: Generalized measurements where the measurement operators are constrained to be orthogonal projectors are the same as projective measurements.
- Exercise: M_m are orthogonal projectors if and only if M_m are Hermitian and $M_m M_{m'} = \delta_{m,m'} M_m$.

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- Exercise: M_m are orthogonal projectors if and only if M_m are Hermitian and $M_m M_{m'} = \delta_{m,m'} M_m$.
- Observation: Generalized measurements where the measurement operators M_m have additional constraints that M_m are Hermitian and $M_m M_{m'} = \delta_{m,m'} M_m$, are the same as projective measurements.

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- Claim: The average value of the measurement, denoted by $\mathbf{E}[M]$, is given by $\mathbf{E}[M] = \langle\psi| M |\psi\rangle$.

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- Exercise: Suppose we measure state ψ that is an eigenvector corresponding to eigenvalue m of the observable M. What is $\mathbf{E}[M]$?

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- Describing the observable M is one way to define the projective measurement. Other ways include:
  - A set of orthogonal projectors P_m satisfying completeness, that is, $\sum_m P_m = I$. The observable in this case is $\sum_m mP_m$.
  - An orthonormal basis $|m\rangle$ in which case, $P_m = |m\rangle \langle m|$.

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- Given that m is the outcome of the measurement, the post-measurement state is $\frac{P_m|\psi\rangle}{\sqrt{p(m)}}$.

- Exercise: Discuss projective measurement of the state $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ w.r.t. observable Z.

- The measurement postulate defines rules for
  1. measurement statistics, and
  2. post-measurement state.
- For certain applications, the post-measurement state is not very important.
  - Can you think of such a scenario?
- POVM stands for Positive Operator-Valued Measure. The main ideas are captured in the following points:
  - For generalised measurement operators M_m and state $|\psi\rangle$, the measurement statistics are given by $p(m) = \langle\psi| M_m^\dagger M |\psi\rangle$.
  - Since we are interested only in the measurement statistics, it will be sufficient to describe the measurement using positive operators

$$E_m \equiv M_m^\dagger M_m$$

  - Observation: $\sum_m E_m = I$ and $p(m) = \langle\psi| E_m |\psi\rangle$.
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- Exercise: Let E_m be an arbitrary set of positive operators such that $\sum_m E_m = I$. Does there exist measurement operators M_m with the same measurement statistics are ones defined by E_m?
  - Yes. $M_m = \sqrt{E_m}$.

- POVM application: Show that the following POVM

$$
\begin{aligned}
E_1 &\equiv \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle \langle 1| \\
E_2 &\equiv \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2} \\
E_3 &\equiv I - E_1 - E_2
\end{aligned}
$$

helps to distinguish states $|0\rangle$ and $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ with the caveat that sometimes it may output "don't know".

Postulate 4

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_n\rangle$.

- We commented earlier that projective measurement is not a weaker notion when compared with generalised measurements (even though it may seem so).
- We will not argue that (Projective measurement + Unitary operators) has the same power generalised measurement.

Lemma

Suppose V is a Hilbert space with a subspace W. Suppose $U : W \rightarrow V$ is a linear operator that preserves inner products, that is, for any $|w_1\rangle, |w_2\rangle \in W$,

$$\langle w_1| U^\dagger U |w_2\rangle = \langle w_1|w_2\rangle .$$

Show that there exists a unitary operator $U' : V \rightarrow V$ that extends U. That is, $U' |w\rangle = U |w\rangle$ for all $|w\rangle \in W$ but U' is defined on the entire space V.

End