COL863: Quantum Computation and Information

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Quantum Mechanics: Linear Algebra

- Outer product: Let $|v\rangle$ be a vector in an inner product space V and $|w\rangle$ be a vector in the inner product space W. $|w\rangle \langle v|$ is a linear operator from V to W defined as:

$$(|w\rangle \langle v|)(|v'\rangle) \equiv |w\rangle \langle v|v'\rangle = \langle v|v'\rangle |w\rangle .$$

- $\sum_i a_i |w_i\rangle \langle v_i|$ is a linear operator which acts on $|v'\rangle$ to produce $\sum_i a_i |w_i\rangle \langle v_i|v'\rangle$.
- Completeness relation: Let $|i\rangle$'s denote orthonormal basis for an inner product space V. Then $\sum_i |i\rangle \langle i| = I$ (the identity operator on V).
- Claim: Let $|v_i\rangle$'s denote the orthonormal basis for V and $|w_j\rangle$'s denote orthonormal basis for W. Then any linear operator $A : V \rightarrow W$ can be expressed in the outer product form as: $A = \sum_{ij} \langle w_j| A |v_i\rangle |w_j\rangle \langle v_i|$.

Cauchy-Schwarz inequality

For any two vectors $|v\rangle , |w\rangle$, $| \langle v|w\rangle |^2 \leq \langle v|v\rangle \langle w|w\rangle$.

- Eigenvector: A eigenvector of a linear operator A on a vector space is a non-zero vector $|v\rangle$ such that $A|v\rangle = v|v\rangle$, where v is a complex number known as the eigenvalue of A corresponding to the eigenvector $|v\rangle$.
- Characteristic function: This is defined to be $c(\lambda) \equiv det(A - \lambda I)$, where *det* denotes determinant for matrices.
  - Fact: The characteristic function depends only on the operator A and not the specific matrix representation for A.
  - Fact: The solution of the characteristic equation $c(\lambda) = 0$ are the eigenvalues of the operator.
  - Fact: Every operator has at least one eigenvalue.
- Eigenspace: The set of all eigenvectors that have eigenvalue v form the eigenspace corresponding to eigenvalue v. It is a vector subspace.
- Diagonal representation: The diagonal representation of an operator A on vector space V is given by $A = \sum_i \lambda_i |i\rangle \langle i|$, where the vectors $|i\rangle$ form an orthonormal set of eigenvectors for A with corresponding eigenvalue λ_i.
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- Degenerate: When an eigenspace has more than one dimension, it is called degenerate. Consider the eigenspace corresponding to eigenvalue 2 in the following example:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Adjoint or Hermitian conjugate: For any linear operator A on vector space V, there exists a unique linear operator $A^\dagger$ on V such that for all vectors $|v\rangle, |w\rangle \in V$:

$$(|v\rangle, A|w\rangle) = (A^\dagger |v\rangle, |w\rangle)$$

Such a linear operator $A^\dagger$ is called the adjoint or Hermitian conjugate of A.

  - Exercise: Show that $(AB)^\dagger = B^\dagger A^\dagger$.
  - By convention, we define $|v\rangle^\dagger \equiv \langle v|$.
  - Exercise: Show that $(A|v\rangle)^\dagger = \langle v| A^\dagger$.
  - Exercise: Show that $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$.
  - Exercise: $(\sum_i a_i A_i)^\dagger = \sum_i a_i^* A_i^\dagger$.
  - Exercise: Show that $(A^\dagger)^\dagger = A$.
  - Exercise: Show that in matrix representation, $A^\dagger = (A^*)^T$.

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- Hermitian or self-adjoint: An operator A with $A^\dagger = A$ is called Hermitian or self-adjoint.
- Projectors: Let W be a k-dimensional vector subspace of a d-dimensional vector space V. There is an orthonormal basis $|1\rangle, ..., |d\rangle$ for V such that $|1\rangle, ..., |k\rangle$ is an orthonormal basis for W. The projector onto the subspace W is defined as:

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    - Observation: The definition is independent of the orthonormal basis used for W.
    - Exercise: Projector P is Hermitian. That is $P^\dagger = P$.
    - Notation: We use vector space P as a shorthand for the vector space onto which P is a projector.
    - Exercise: Show that for any projector $P^2 = P$.
- Orthogonal complement: The orthogonal complement of a projector P is the operator $Q \equiv I - P$.
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- Normal operator: An operator A is said to be normal if $AA^\dagger = A^\dagger A$.

Spectral Decomposition Theorem

Any normal operator M on a vector space V is a diagonalizable with respect to some orthonormal basis for V. Conversely, any diagononalizable operator is normal.

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- Exercise: Show that a normal matrix is Hermitian if and only if it has real eigenvalues.
- Unitary matrix: A matrix U is called unitary if $UU^\dagger = U^\dagger U = I$.
- Unitary operator: An operator U is unitary if $UU^\dagger = U^\dagger U = I$.
- Exercise: Show that unitary operators preserve inner products.
- Exercise: Let $|v_i\rangle$ be any orthonormal basis set and let $|w_i\rangle = U|v_i\rangle$. Then $|w_i\rangle$ is an orthonormal basis set. Moreover, $U = \sum_i |w_i\rangle\langle v_i|$.
- Exercise: If $|v_i\rangle$ and $|w_i\rangle$ are two orthonormal basis sets, then $U \equiv \sum_i |w_i\rangle\langle v_i|$ is a unitary operator.
- Exercise: Show that all the eigenvalues of a unitary matrix have modulus 1. This means that they can be written as $e^{i\theta}$ for some real θ.

End