COL863: Quantum Computation and Information

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Quantum Mechanics: Linear Algebra

- Linear algebra: Study of vector spaces and linear operations on those vector spaces.
- The quantum mechanical notation of a vector in a vector space is $|\psi\rangle$, where ψ is the label for the vector.
- The zero vector of the vector space is denoted using $\mathbf{0}$. We do not use $|0\rangle$ since this is used to denote something else.
- A spanning set for a vector space is a set of vectors $|v_1\rangle, ..., |v_n\rangle$ such that any vector of the vector space can be written as a linear combination $|v\rangle = \sum_i a_i |v_i\rangle$.

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$$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad |v_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

  - Question: Express $\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right]$ as a combination of $|v_1\rangle$ and $|v_2\rangle$.

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- A set of non-zero vectors is linearly dependent if there exists a set of complex numbers $a_1, ..., a_n$ with $a_i \neq 0$ for at least one value of i such that

$$a_1 |v_1\rangle + ... + a_n |v_n\rangle = \mathbf{0}$$

  A set of vectors is linearly independent if it is not linearly dependent.
  - Question: Are the vectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ linearly dependent?

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A set of vectors is linearly independent if it is not linearly dependent.
- Fact: Any two sets of linearly independent spanning sets contain the same number of vectors. Any such set is called a basis for the vector space. Moreover, such a basis set always exists.
- The number of elements in any basis is called the dimension of the vector space.
- In this course, we will only be interested in *finite dimensional* vector spaces.

- A linear operator between vector spaces V and W is defined to be any function $A : V \rightarrow W$ that is linear in its input:

$$A\left(\sum_i a_i \left|v_i\right\rangle\right) = \sum_i a_i A \left|v_i\right\rangle .$$

(We use $A\left|.\right\rangle$ in short to indicate $A(\left|.\right\rangle)$). A linear operator on a vector space V means that the linear operator is from V to V.

  - Example: Identity operator I_V on any vector space V satisfies $I_v \left|v\right\rangle = \left|v\right\rangle$ for all $\left|v\right\rangle \in V$.
  - Example: Zero operator 0 on any vector space V satisfies $0\left|v\right\rangle = \mathbf{0}$ for all $\left|v\right\rangle \in V$.
  - Claim: The action of a linear operator is completely determined by its action on the basis.

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- Composition: Given vector spaces V, W, X and linear operators $A : V \rightarrow W$ and $B : W \rightarrow X$, then BA denotes the linear operator from V to X that is a composition of operators B and A. We use $BA|v\rangle$ to denote $B(A(|v\rangle))$.

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- Matrix representation: Let $A : V \rightarrow W$ be a linear operator and let $\left|v_1\right\rangle, ..., \left|v_m\right\rangle$ be basis for V and $\left|w_1\right\rangle, ..., \left|w_n\right\rangle$ be basis for W. Then for every $1 \leq j \leq m$, there are complex numbers $A_{1j}, ..., A_{nj}$ such that

$$A\left|v_j\right\rangle = \sum_i A_{ij}\left|w_i\right\rangle.$$

  - Question: Let V be a vector space with basis $\left|0\right\rangle, \left|1\right\rangle$ and $A : V \rightarrow V$ be a linear operator such that $A\left|0\right\rangle = \left|1\right\rangle$ and $A\left|1\right\rangle = \left|0\right\rangle$. Give the matrix representation of A.

- Inner product: Inner product is a function that takes two vectors and produces a complex number (denoted by $(.,.)$).
- A function $(.,.)$ from $V \times V \rightarrow \mathbb{C}$ is an inner product if it satisfies the requirement that:
  1. $(.,.)$ is linear in the second argument. That is
  $$\left(|v\rangle, \sum_i \lambda_i |w_i\rangle\right) = \sum_i \lambda_i (|v\rangle, |w_i\rangle).$$
  2. $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$.
  3. $(|v\rangle, |v\rangle) \geq 0$ with equality if and only if $|v\rangle = 0$.
- Question: Show that $(\sum_i \lambda_i |w_i\rangle, |v\rangle) = \sum_i \lambda_i^* (|w_i\rangle, |v\rangle)$.

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  3. $(|v\rangle, |v\rangle) \geq 0$ with equality if and only if $|v\rangle = 0$.
- Inner Product Space: A vector space equipped with an inner product is called an inner product space.
- In finite dimensions, a Hilbert space is simply an inner product space.

- Dual vector: $\langle v|$ is used to denote the dual vector to the vector $|v\rangle$. The dual is a linear operator from an inner product space V to complex number $\mathbb{C}$, defined by $\langle v|\left(|w\rangle\right) \equiv \langle v|w\rangle \equiv \left(|v\rangle, |w\rangle\right)$.
- Orthogonal: Vectors $|w\rangle$ and $|v\rangle$ are orthogonal if their inner product is 0.
- Norm: The norm of a vector $|v\rangle$ denoted by $|| \, |v\rangle \, ||$ is defined as:

$$|| \, |v\rangle \, || = \sqrt{\langle v|v\rangle}$$

- Unit vector: A unit vector is a vector $|v\rangle$ such that $|| \, |v\rangle \, || = 1$.
- Normalized vector: $\frac{|v\rangle}{|| |v\rangle ||}$ is called the normalized form of vector $|v\rangle$.
- Orthonormal set: A set of vectors $|1\rangle, ..., |n\rangle$ is orthonormal if each vector is a unit vector and distinct vectors in the set are orthogonal. That is $\langle i|j\rangle = \delta_{ij}$.

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- Let $|w_1\rangle, ..., |w_d\rangle$ be a basis set for some inner product space V. The following method, called the Gram-Schmidt procedure, produces an orthonormal basis set $|v_1\rangle, ..., |v_d\rangle$ for the vector space V.

Gram-Schmidt procedure

- $|v_1\rangle = \frac{|w_1\rangle}{|||w_1\rangle||}$.
- For $1 \leq k \leq d-1$, $|v_{k+1}\rangle$ is inductively defined as:

$$|v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i|w_{k+1}\rangle |v_i\rangle}{|| \, |w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i|w_{k+1}\rangle |v_i\rangle \, ||}$$

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- Question Show that the Gram-Schmidt procedure produces an orthonormal basis for V.

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- Theorem: Any finite dimensional inner product space of dimension d has an orthonormal basis $|v_1\rangle, ..., |v_d\rangle$.

- Orthonormal set: A set of vectors $|1\rangle, ..., |n\rangle$ is orthonormal if each vector is a unit vector and distinct vectors in the set are orthogonal. That is $\langle i|j\rangle = \delta_{ij}$.
- Consider an orthonormal basis $|1\rangle, ..., |n\rangle$ for an inner product space V. Let $|v\rangle = \sum_i v_i |i\rangle$ and $|w\rangle = \sum_i w_i |i\rangle$. Then

$$\langle v|w\rangle = \left(\sum_i v_i |i\rangle, \sum_j w_j |j\rangle\right) =?$$

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$$\langle v|w\rangle = \left(\sum_i v_i |i\rangle, \sum_j w_j |j\rangle\right) = \sum_{ij} v_i^* w_j \delta_{ij} = \begin{bmatrix} v_1^* & \dots & v_n^* \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

- Dual vector $\langle v|$ has a row vector representation as seen above.

- Outer product: Let $|v\rangle$ be a vector in an inner product space V and $|w\rangle$ be a vector in the inner product space W. $|w\rangle\langle v|$ is a linear operator from V to W defined as:

$$(|w\rangle\langle v|)(|v'\rangle) \equiv |w\rangle\langle v|v'\rangle = \langle v|v'\rangle |w\rangle .$$

- $\sum_i a_i |w_i\rangle\langle v_i|$ is a linear operator which acts on $|v'\rangle$ to produce $\sum_i a_i |w_i\rangle\langle v_i|v'\rangle$.
- Completeness relation: Let $|i\rangle$'s denote orthonormal basis for an inner product space V. Then $\sum_i |i\rangle\langle i| = I$ (the identity operator on V).
- Claim: Let $|v_i\rangle$'s denote the orthonormal basis for V and $|w_j\rangle$'s denote orthonormal basis for W. Then any linear operator $A : V \rightarrow W$ can be expressed in the outer product form as:

$$A = \sum_{ij} \langle w_j| A |v_i\rangle |w_j\rangle\langle v_i|$$

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Cauchy-Schwarz inequality

For any two vectors $|v\rangle, |w\rangle$, $|\langle v|w\rangle|^2 \leq \langle v|v\rangle\langle w|w\rangle$.

End