COL863: Quantum Computation and Information

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Introduction

- Multiple qubit gates:
  - Claim: We saw that there is a quantum analogue of the classical NOT gate. If there is a similar analogue for NAND gate, then any classical logic circuit will have a quantum analogue.
  - Why should the above claim hold? NAND gate is a universal gate.
  - Does a quantum analogue of NAND gate exist? No
    - NAND gate is irreversible. That is one cannot obtain A and B from $A \wedge B$.
    - Quantum gates are constrained to be reversible.
    - Unitary gates (operations using unitary matrices) are invertible and hence reversible.

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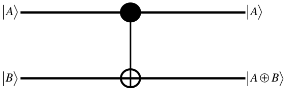
  - Is there a reversible gate that is universal for quantum computation? Yes
    - This is called the controlled-NOT gate or CNOT gate.



    - More precisely, the matrix representing the gate is given by

$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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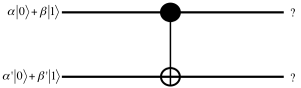

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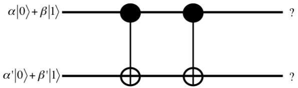

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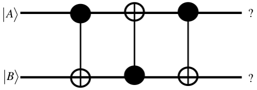

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    - Claim: *Any multiple qubit logic gate may be composed from* CNOT *and single qubit gates.*

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- Measurements:
  - We now have a high-level understanding of how a quantum circuit evolves. What can be obtain or measure from the circuit?
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    - We can measure in any orthonormal basis $|a\rangle, |b\rangle$. If the state of the qubit can be expressed as $\alpha |a\rangle + \beta |b\rangle$, then the measurement result is a with probability $|\alpha|^2$ and b with probability $|\beta|^2$.
    - One such popular basis is the $|+\rangle, |-\rangle$ basis that are expressed as $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$.
    - Question: Express $\alpha |0\rangle + \beta |1\rangle$ in the $|+\rangle, |-\rangle$ basis.

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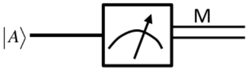
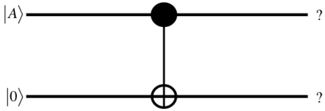
    - In quantum circuit diagrams, measurement of a qubit is represented as below:



- Some exercises:
  - What is the output of the following circuit?



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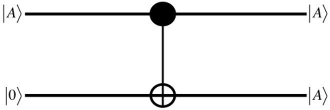

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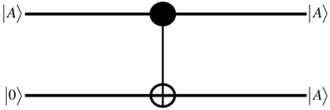

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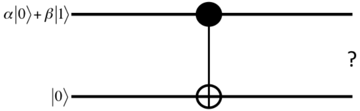
    - Consider what happens in the following circuit?



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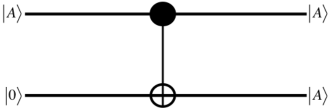

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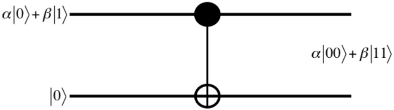

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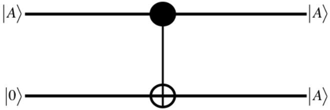


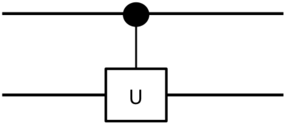
  - So, is the above circuit a qubit-copying circuit? No
  - No-Cloning Theorem: It is impossible to copy an unknown quantum state input.

- Some exercises:
  - Let $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ be any unitary matrix representing a single-qubit gate Q. Consider the matrix:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p & q \\ 0 & 0 & r & s \end{bmatrix}$$

    Is this matrix unitary?

- Some exercises:
  - Let $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ be any unitary matrix representing a single-qubit gate U. Consider the matrix:
    $$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p & q \\ 0 & 0 & r & s \end{bmatrix}$$
    Is this matrix unitary? Yes
  - So, this is a valid two-qubit quantum gate and is represented as:



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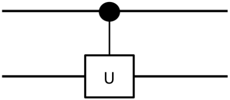
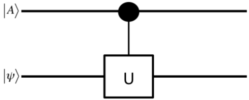


  - Question: Draw the quantum truth table for the circuit below:



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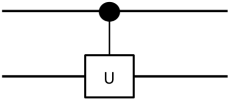

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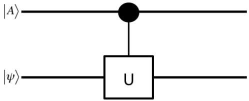

Input qubits	Output qubits
$\vert 0\rangle \vert \psi\rangle$	$\vert 0\rangle \vert \psi\rangle$
$\vert 1\rangle \vert \psi\rangle$	$\vert 1\rangle\, U(\vert \psi\rangle)$

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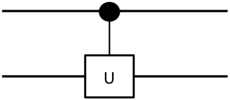

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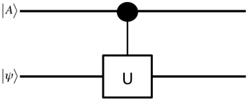
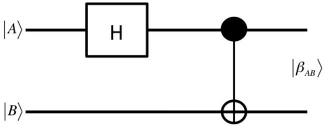


Input qubits	Output qubits
$\|0\rangle \|\psi\rangle$	$\|0\rangle \|\psi\rangle$
$\|1\rangle \|\psi\rangle$	$\|1\rangle U(\|\psi\rangle)$

  - This is known as the controlled-U gate. The U gate is conditionally applied to the second qubit.

- Some exercises:
  - What is the output of the following circuit for different input states as shown:



In	Out
$\|00\rangle$	?
$\|01\rangle$	?
$\|10\rangle$	?
$\|11\rangle$	?

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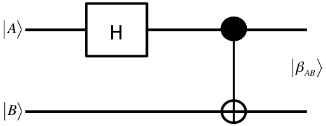
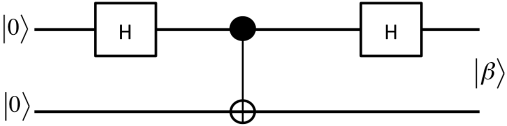


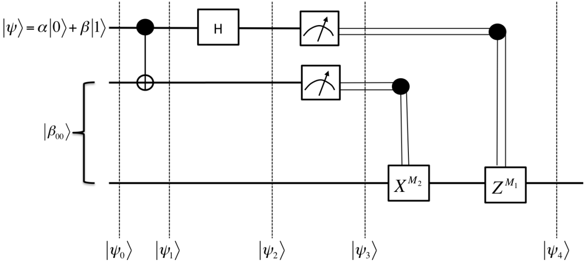
In	Out
$\|00\rangle$	$\frac{\|00\rangle+\|11\rangle}{\sqrt{2}} \equiv \|\beta_{00}\rangle$
$\|01\rangle$	$\frac{\|01\rangle+\|10\rangle}{\sqrt{2}} \equiv \|\beta_{01}\rangle$
$\|10\rangle$	$\frac{\|00\rangle-\|11\rangle}{\sqrt{2}} \equiv \|\beta_{10}\rangle$
$\|11\rangle$	$\frac{\|01\rangle-\|10\rangle}{\sqrt{2}} \equiv \|\beta_{11}\rangle$

  - $|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle$ are called Bell states or EPR-pairs or EPR-states (after Bell, Einstein, Podolsky, and Rosen). These exhibit interesting properties as we will see in our first application to quantum-teleportation.

- Some exercises:
  - What is the output of the following circuit:



- Alice and Bob met sometime back and together they created Bell pair $|\beta_{00}\rangle$ and both kept one qubit each.
- They are now very far from each other perhaps in some opposite corners of the universe.
- Alice wants to deliver an unknown qubit $|\psi\rangle$ to Bob. Moreover, she can only communicate classical information to Bob.
- Fortunately, she knows quantum circuits and constructs the following circuit in a hope to communicate $|\psi\rangle$. The first two qubits in the circuit is in possession of Alice while Bob has the third qubit.



End