Name: ______________________________

Entry number: ________________________

There are 3 questions for a total of 20 points.

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1. (6 points) Find the eigenvalues and corresponding eigenvectors for the following matrices (corresponding to single qubit gates):

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

> **Solution:**
>
> - Matrix X: The characteristic equation corresponding to X is $\lambda^2 - 1 = 0$ that gives $\lambda = \pm 1$. The eigenvectors for $+1, -1$ solves to $\frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $\frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$ respectively.
> - Matrix Y: Eigenvalues $+1, -1$ with eigenvectors $\frac{1}{\sqrt{2}} \left[\begin{smallmatrix} -i \\ 1 \end{smallmatrix}\right]$ and $\frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]$ respectively.
> - Matrix Z: Eigenvalues $+1, -1$ with eigenvectors $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ respectively.
> - Matrix S: Eigenvalues $1, i$ with eigenvectors $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ respectively.
> - Matrix H: Eigenvalues $+1, -1$ with eigenvectors $\left[\begin{smallmatrix} 1 \\ -1+\sqrt{2} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ -1-\sqrt{2} \end{smallmatrix}\right]$ respectively.

2. (6 points) Suppose Bob is given a quantum state chosen from a set $|\psi_1\rangle, |\psi_2\rangle, ..., |\psi_m\rangle$ of linearly independent states. Construct a POVM $\{E_1, E_2, ..., E_{m+1}\}$ such that if outcome E_i occurs, $1 \leq i \leq m$, then Bob knows with certainty that he was given the state $|\psi_i\rangle$.

> **Solution:** For every state $|\psi_i\rangle$, we will construct a state $|\psi_i'\rangle$ with the property that:
>
> $$\forall j \neq i, \langle \psi_j | \psi_i' \rangle = 0. \tag{1}$$
>
> Then, we will set the POVM as: $E_i = \frac{1}{m} |\psi_i'\rangle \langle \psi_i'|$ for $i = 1, ..., m$ and $E_{m+1} = I - \sum_{i=1}^{m} E_i$. From (1) it follows that if E_i is the outcome, then the probability that the pre-measurement state was $|\psi_j\rangle$ for some $j \neq i$ is 0. We will now argue that $E_1, ..., E_{m+1}$ are valid POVM elements. For this, we need to show that:
> 1. E_i's are positive
> 2. $E_1 + ... + E_{m+1} = I$.
>
> The second condition follows from the definition. The fact that $E_1, ..., E_m$ are positive follows from the fact that E_i is an outer product. For E_{m+1}, consider any vector $|v\rangle$:
>
> $$\begin{aligned} \langle v| E_{m+1} |v\rangle &= \langle v| I |v\rangle - \sum_{i=1}^{m} \langle v| E_i |v\rangle \\ &= || \, |v\rangle \, ||^2 - \sum_{i=1}^{m} \frac{1}{m} \langle v| |\psi_i'\rangle \langle \psi_i'| |v\rangle \\ &\geq 0 \quad \text{(Since for any vector } |v\rangle \text{ and state } |\psi\rangle, \langle v| |\psi_i'\rangle \langle \psi_i'| |v\rangle \leq || \, |v\rangle \, ||^2 \text{)} \end{aligned}$$

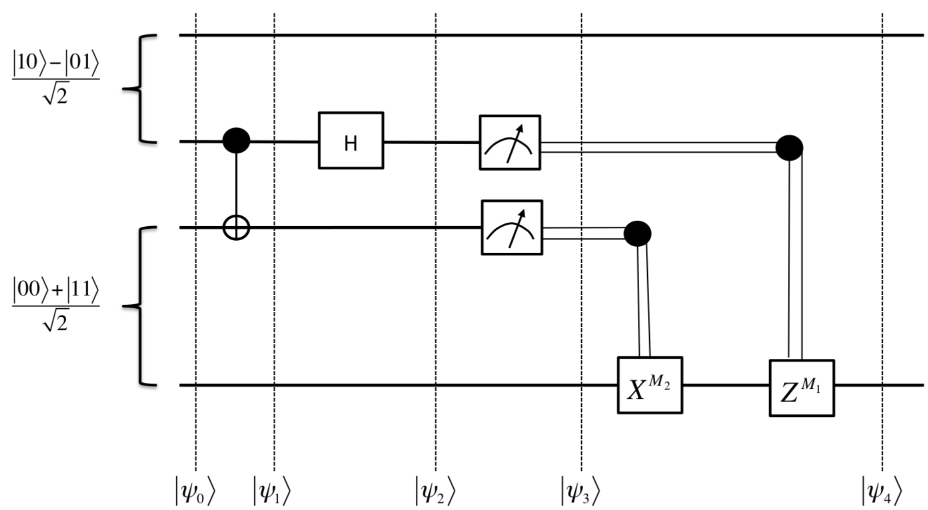
What remains is to show the construction of the states $|\psi_i'\rangle$ such that eqn. (1) holds. For $i = 1, ..., m$, let P_i denote the projector for the space spanned by the vectors $\{|\psi_1\rangle, ..., |\psi_m\rangle\} \setminus |\psi_i\rangle$. Then define:

$$|\psi_i''\rangle = |\psi_i\rangle - P_i |\psi_i\rangle \quad \text{and} \quad |\psi_i'\rangle = \frac{|\psi_i''\rangle}{|| \, |\psi_i''\rangle \, ||}.$$

Note that for any $j \neq i$, we have $\langle \psi_j | \psi_i'' \rangle = 0$ which is exactly what we wanted.

3. (8 points) Suppose you have two qubits in the bell state $\frac{|01\rangle - |10\rangle}{\sqrt{2}}$ and you apply the teleportation protocol to the first qubit. What is the result? (*Please try giving an appropriate interpretation for your calculations.*)

Solution: The teleportation scenario is described in the following figure:



We can calculate the states of the above system.

$$
\begin{aligned}
|\psi_0\rangle &= \left(\frac{|10\rangle - |01\rangle}{\sqrt{2}}\right)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) \\
&= \frac{1}{2}\left(|1000\rangle + |1011\rangle - |0100\rangle - |0111\rangle\right) \\
|\psi_1\rangle &= \frac{1}{2}\left(|1000\rangle + |1011\rangle - |0110\rangle - |0101\rangle\right) \\
|\psi_2\rangle &= \frac{1}{2}\left(|1\rangle\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|00\rangle + |1\rangle\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|11\rangle - |0\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)|10\rangle - |0\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)|01\rangle\right)
\end{aligned}
$$

Since a measurement happens after state $|\psi_2\rangle$, $|\psi_3\rangle$ is the following ensemble of states:

$$|\psi_3\rangle = \begin{cases} \frac{|1000\rangle - |0001\rangle}{\sqrt{2}} & \text{w.p. } 1/4 \\ \frac{|1011\rangle - |0010\rangle}{\sqrt{2}} & \text{w.p. } 1/4 \\ \frac{|1100\rangle + |0101\rangle}{\sqrt{2}} & \text{w.p. } 1/4 \\ \frac{|1111\rangle + |0110\rangle}{\sqrt{2}} & \text{w.p. } 1/4 \end{cases}$$

In the first case, the X and Z gates are not applied to the last qubit and hence the final state of the first and the last qubit is $|\psi_4\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$.
In the second case, only the X gate is applied to the last qubit and hence the final state of the first and the last qubit is $|\psi_4\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$.
In the third case, the Z gate is applied to the last qubit and hence the final state of the first and the last qubit is $|\psi_4\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$.
In the fourth case, the X and Z gate is applied to the last qubit and hence the final state of the first and the last qubit is $|\psi_4\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$.

So, $|\psi_4\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$. Note that this is the same state as the qubits that Alice had in the beginning. So, effectively the **entanglement has been teleported** in this protocol.