Name: ______________________

Entry number: ______________________

There are 5 questions for a total of 35 points.

---

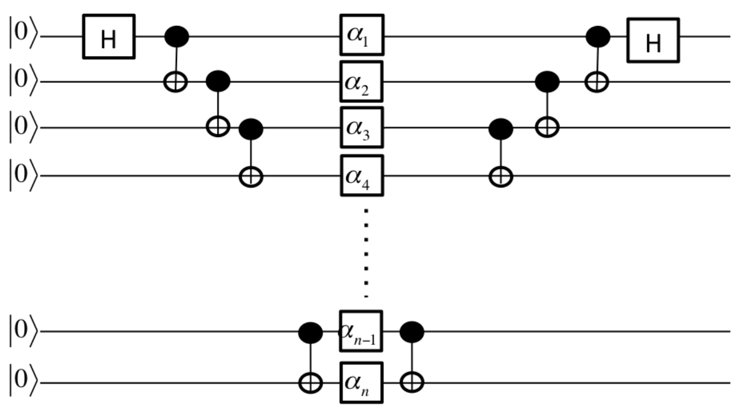
1. Answer the following questions:

   (a) (1 ½ points) Draw a quantum circuit on two qubits that copies the $|+\rangle, |-\rangle$ state. That is, the circuit should have the following behaviour:

Input	Output
$\|+\rangle\|0\rangle$	$\|+\rangle\|+\rangle$
$\|-\rangle\|0\rangle$	$\|-\rangle\|-\rangle$

   (b) (1 ½ points) Draw a 4-qubit quantum circuit that takes the state $|0000\rangle$ to $\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$.

   (c) (2 points) Consider the circuit below:

   Here α_i denotes the operation $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha_i} \end{bmatrix}$. Suppose at the end the measurement is taken in the computational basis. What is the probability of seeing $|00...0\rangle$?

   (d) (3 points) Consider the Shor's 9-qubit error correcting code discussed in the class. Suppose the 9-qubit code for $\alpha|0\rangle + \beta|1\rangle$ undergoes the following changes: the second and sixth qubits get bit-flipped and the first and eighth qubit get phase-flipped. Show the intermediate states of the 9 qubits in the process of error correction.

2. (5 points) For $p, q \in \{0,1\}$ the 3-qubit unitary operation $U_{p,q}$ is defined as:

$$U_{p,q}|x,y,z\rangle = \begin{cases} |xy\rangle|z \oplus 1\rangle & \text{if } x = p \text{ and } y = q \\ |xy\rangle|z\rangle & \text{Otherwise} \end{cases}$$

   Starting with a 3-qubit input state $|001\rangle$ consider the following sequence of operations:

   1. Apply $H \otimes H \otimes H$
   2. Apply $U_{a,b}$ for some $a, b \in \{0,1\}$
   3. Apply $H \otimes H \otimes I$
   4. Apply $U_{0,0}$
   5. Apply $H \otimes H \otimes H$
   6. Measure the 3 qubits in computational basis

   Describe the measurement output. You may write your answer in terms of a and b. Give appropriate explanation.

3. Consider the following 3-person game defined similarly to the 2-person CHSH game discussed in the class:

   > Alice, Bob, and Dave receive randomly generated bits $x, y, z \in \{0,1\}$ such that $x \oplus y \oplus z = 0$ from Charlie. Their goal is to respond with bits a, b, and c such that $a \oplus b \oplus c = x \vee y \vee z$. They are not allowed to communicate after receiving x, y, z.

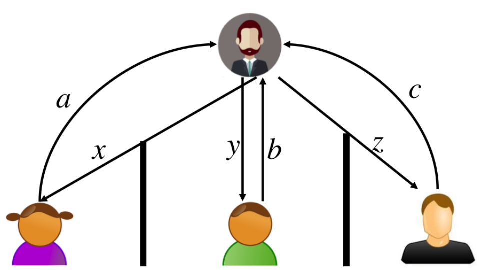


Figure 1: The 3-person game.

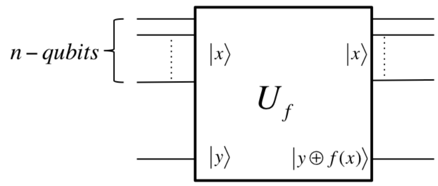
   Prove or disprove:

   (a) (3 points) There is no classical deterministic or randomized strategy that allows Alice, Bob, and Dave to win with probability more than 3/4.

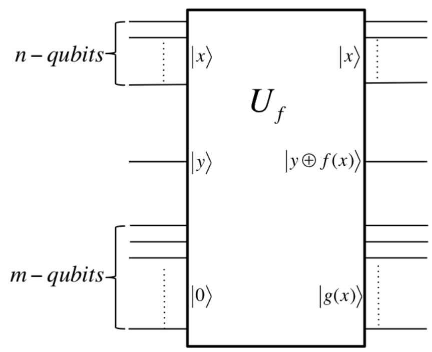
   (b) (5 points) There is a quantum strategy that allows Alice, Bob, and Dave to win with probability 1. Assume that Alice, Bob, and Dave share the qubits in the entangled state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

4. (8 points) Let $f : \{0,1\}^n \to \{0,1\}^n$ be a 2-to-1 function. Design a Quantum algorithm for finding a *collision* (that is, $x \neq y$ such that $f(x) = f(y)$) with query complexity $O(n \cdot 2^{n/3})$.
   (Hint: Use a combination of classical and quantum ideas.)

5. (6 points) In the lectures and in the exams we have used the following diagram for unitary operation with respect to a boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$:



We noted that this was a simplification and does not represent the correct scenario since evaluation of the function f may involve additional ancilla qubits. So, the correct picture (taking into consideration the ancilla qubits) is given below.



The m ancilla qubits are initialised to $|0\rangle$ and the behaviour of this quantum gate in general is given as:

$$U_f \, |x\rangle \, |y\rangle \, |0\rangle = |x\rangle \, |y \oplus f(x)\rangle \, |g(x)\rangle$$

Here $g(x)$ is some arbitrary function depending on the input x. We said in the class that it is a good idea to "uncompute" the ancilla qubits. That is, make sure that $g(x) = 0$. Note that this is always possible because of reversible nature of quantum gates. Suppose, U_f does not "uncompute" the ancilla qubits which means that $g(x)$ may be some non-zero function. Discuss what could go wrong in the context of the Deutsch-Jozsa algorithm. The circuit for the Deutsch-Jozsa algorithm is given below.