CSL202: Discrete Mathematical Structures
Tutorial/Homework: 08

1. Suppose that 21 girls and 21 boys enter a mathematics competition. Furthermore, suppose that each entrant solves at most six questions, and for every boy-girl pair, there is at least one question that they both solved. Show that there is a question that was solved by at least three girls and at least three boys.

2. How many ways are there for a horse race with four horses to finish if ties are possible?

3. Prove the *Hockeystick identity*

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

   whenever n and r are positive integers,

   (a) using a combinatorial argument.

   (b) using Pascal's identity.

4. Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

5. Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$.

6. Suppose that a weapons inspector must inspect each of five different sites twice, visiting one site per day. The inspector is free to select the order in which to visit these sites, but cannot visit site X, the most suspicious site, on two consecutive days. In how many different orders can the inspector visit these sites?

7. Consider the following two theorems:

   Theorem 1.0.1 (Permutation with indistinguishable objects) *The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type $2, ...,$ and n_k indistinguishable objects of type k, is*

$$\frac{n!}{n_1!n_2!...n_k!}.$$

   Theorem 1.0.2 (Distinguishable objects into distinguishable boxes) *The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, ..., k$, equals*

$$\frac{n!}{n_1!n_2!...n_k!}$$

Prove the second theorem by first setting up a one-to-one correspondence between permutations of n objects with n_i indistinguishable objects of type $i, i = 1, 2, ..., k$ and the distribution of n objects in k boxes such that n_i objects are placed in box $i, i = 1, 2, ..., k$ and then applying the first theorem.

8. Six swimmers training together either swam in a race or watched the others swim. At least how many races must have been scheduled if every swimmer had opportunity to watch all of the others?

9. Suppose that instead of three doors, there are four doors in the Monty Hall puzzle. What is the probability that you win by not changing once the host, who knows what is behind each door, opens a losing door and gives you the chance to change doors? What is the probability that you win by changing the door you select to one of the two remaining doors among the three that you did not select?

10. Which is more likely: rolling a total of 8 when two dice are rolled or rolling a total of 8 when three dice are rolled?

11. What is the probability of these events when we randomly select a permutation of $\{1, 2, ..., n\}$ where $n \geq 4$?

    a) 1 precedes 2.
    b) 2 precedes 1.
    c) 1 immediately precedes 2.
    d) n precedes 1 and n-1 precedes 2.
    c) n precedes 1 and n precedes 2.

12. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?