CSL202: Discrete Mathematical Structures

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Relations

- Relation are mathematical structures used to represent relationships between elements of sets.
- These are just subset of cartesian product of sets.

Definition (Binary relation)

Let A and B be sets. A *binary relation* from A to B is a subset of $A \times B$.

- We use $a\ R\ b$ to denote $(a, b) \in R$ and $a \not R b$ to denote $(A, b) \notin R$.

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- Example: Let A be the set of cities and B be the set of states. Consider the relation R denoting "is in state". So, $(a, b) \in R$ iff city a is in state b. So, $(Lucknow, UP) \in R$.

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  - 2^{n^2}

Definition (Reflexive)

A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

Definition (Symmetric and antisymmetric)

A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called antisymmetric.

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A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

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- Question: Is the "divides" relation on the set of positive integers transitive?
- Question: How many reflexive relations are there on a set with n elements?

- Since relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.
- Question: Let R_1 be the "less than" relation on the set of real numbers and let R_2 be the "greater than" relation on the set of real numbers. What are:
  1. $R_1 \cup R_2 = ?$
  2. $R_1 \cap R_2 = ?$
  3. $R_1 - R_2 = ?$
  4. $R_2 - R_1 = ?$
  5. $R_1 \oplus R_2 = ?$

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- Question: Let R_1 be the "less than" relation on the set of real numbers and let R_2 be the "greater than" relation on the set of real numbers. What are:
  1. $R_1 \cup R_2 = \{(x, y) | x \neq y\}$
  2. $R_1 \cap R_2 = \emptyset$
  3. $R_1 - R_2 = R_1$
  4. $R_2 - R_1 = R_2$
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Definition (Composite)

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

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- Question: Let $A = \{1, 2, 3\}, B = \{1, 2, 3, 4\}, C = \{0, 1, 2\}$, $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$, and $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$. What is $S \circ R$?

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- Question: Let A be the set of all people and let R denote the "is parent" relationship. What relationship does $R \circ R$ capture?

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Definition

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, ...$ are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

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Theorem

A relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, ...$

Definition (n-ary relation)

Let $A_1, A_2, ..., A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times ... \times A_n$. The sets $A_1, A_2, ..., A_n$ are called the domains of the relation, and n is called its degree.

- Used in relational databases.

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Representation using Matrices

Consider a relation R from a finite sets $A = \{a_1, ..., a_m\}$ to $B = \{b_1, ..., b_n\}$ (*elements of these sets are listed in a particular but arbitrary order*). The relation R is represented by the matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

- Show that: A relation R is antisymmetric iff for all $i \neq j$, either $m_{ij} = 0$ or $m_{ji} = 0$.

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- Show that: $M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$ and $M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$.

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- <u>Show that</u>: $M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$ and $M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$.
- <u>Question</u>: Find the matrix representing R^2, when the matrix representing R is

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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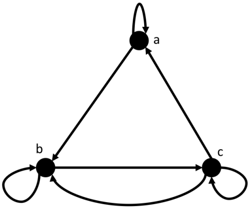
Representation using directed graphs

A directed graph or digraph consists of a set V of vertices (or nodes) together with a set E or ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a, b) and the vertex b is called the terminal vertex of this edge.

- Determine whether the relation for the directed graph shown below is reflexive, symmetric, antisymmetric, and/or transitive.



Closure

A relation S on a set A is called the **closure** of another relation R on A with respect to property P if S has property P, S contains R, and S is a subset of every relation with property P containing R.

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- Question: How do we find the transitive closure of any relation R on set A?
  - Consider a relation $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on set $A = \{1, 2, 3, 4\}$.
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  - There is an immediate need to add $(1, 2), (2, 3), (2, 4), (3, 1)$ for transitivity.
  - Question: Does the resulting relation become transitive after adding the above?

- Question: How do we find the transitive closure of any relation R on set A?

Definition (Path/cycle in directed graph)

A path from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n)$ in G, where n is a non-negative integer, and $x_0 = a$ and $x_n = b$. This path is denoted by $x_0, x_1, ..., x_{n-1}, x_n$ and has a length n. We view the empty set of edges as a path from a to a. A path of length $n \geq 1$ that begins and ends at the same vertex is called a circuit or cycle.

- The concept of path and cycles also applies to relations (since relations can be represented as digraphs).

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Theorem

Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

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Definition (Connectivity relation)

Let R be a relation on a set A. The connectivity relation R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

- Claim: $R^* = \cup_{n=1}^{\infty} R^n$.

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The transitive closure of a relation R equals the connectivity relation R^*.

End