CSL202: Discrete Mathematical Structures

Ragesh Jaiswal, CSE, IIT Delhi

Counting

- What are we going to count?
  - Objects with certain properties.
- Basic Counting Principles:
  - The Product Rule: Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.

- Basic Counting Principles:
  - The Product Rule: Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.
  - Extended Product Rule: Suppose that a procedure is carried out by performing the tasks $T_1, T_2, ..., T_m$ in sequence. If each task $T_i, i = 1, 2, ..., n$, can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 ... n_m$ ways to carry out the procedure.
  - Can you argue the validity of the extended product rule using the product rule?

- Basic Counting Principles:
  - The Product Rule: Examples
    - How many functions are there from a set of m elements to a set of n elements?
    - How many such one-one functions are there?

- Basic Counting Principles:
  - The Product Rule:
  - The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.
  - The Extended Sum Rule: Suppose that a task can be done in one of n_1 ways, in one of n_2 ways,...,or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with$1 \leq i < j \leq m$. Then the number of ways to do the task is $n_1 + n_2 + ... + n_m$.

- Basic Counting Principles:
  - The Product Rule:
  - The Sum Rule: Examples
    - Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

- Basic Counting Principles:
  - The Product Rule:
  - The Sum Rule:
  - The Subtraction Rule: If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
    - The subtraction rule is also known as the principle of *inclusion-exclusion*, especially when it is used to count the number of elements in the union of two sets.
    - $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

- Basic Counting Principles:
  - The Product Rule:
  - The Sum Rule:
  - The Subtraction Rule:
  - The Division Rule: If f is a function from A to B where A and B are finite sets, and that for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$ (in which case, we say that f is d-to-one), then $|B| = |A|/d$.
    - Example: How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

- Basic Counting Principles:
  - The Product Rule
  - The Sum Rule
  - The Subtraction Rule
  - The Division Rule

- Tree diagrams: We use a branch to represent each possible choice. The possible outcomes are represented by the *leaves*.
  - Example: A playoff between two teams consists of at most five games. The first team that wins three games wins the playoff. In how many different ways can the playoff occur?

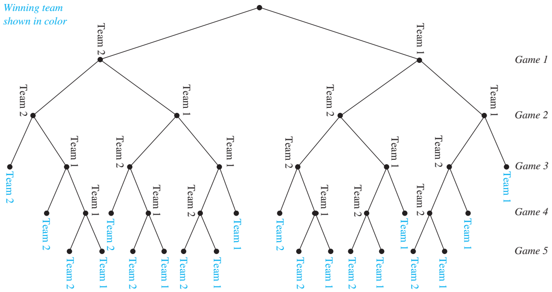

FIGURE 3 **Best Three Games Out of Five Playoffs.**

Theorem (The Pigeonhole Principle)

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Corollary

A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.

Theorem (The Pigeonhole Principle)

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Corollary

A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.

- Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Theorem (The Generalized Pigeonhole Principle)

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Theorem (The Generalized Pigeonhole Principle)

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

- What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

- Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

End