CSL202: Discrete Mathematical Structures

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Warm-up exercise

- You must be familiar with the definition of Big-O from Data Structures. Note that the definition is an interesting example of nested quantifiers.

Definition (Big-O)

Let $f(n)$ and $g(n)$ denote functions mapping positive integers to positive real numbers. The function $f(n)$ is said to be $O(g(n))$ (or $f(n) = O(g(n))$) in short if and only if there exists constants $C, n_0 > 0$ such that for all $n \geq n_0, f(n) \leq C \cdot g(n)$.

- How would you argue that $5n^2 + 3n + 1$ is $0(n^2)$?
- How would you argue that 2^n is not $O(n)$?

Rules of Inference

- Consider the following argument.
  - "*If you have a current password, then you can log onto the network.*"
  - "*You have a current password.*"
  - Therefore "*you can log onto the network.*"
- Is this a valid argument? Why?

- Argument: A sequence of statements that end with a conclusion.
- Valid argument: The conclusion must follow from the truth of the preceding statements (known as *premises*). An argument is valid if and only if it is impossible for all the premises to be true and conclusion to be false.
- Rules of inference: Templates for obtaining new statements from already available statements or in other words templates for constructing valid arguments.

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- Let us write this argument more concisely.
  - Propositions:
    - p: "*You have a current password.*"
    - q: "*You can log onto the network.*"
  - Argument:

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

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- We know that for any propositions p, q, r, $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology.
- So, the initial argument is a valid argument.

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- We know that for any propositions p, q, r, $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology.
- So, the initial argument is a valid argument.
- Moreover, if we plug in any propositions into p and q such that the premises $p \rightarrow q$ and p are true for these propositions, then concluding q from these premises is a valid argument.
- This is called an *argument form*. The validity of an argument follows from the validity of the argument form.

Definition (Argument and argument form)

An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*. An argument is valid if the truth of all its premises implies that the conclusion is true.
An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* if no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

- How do we show that an argument form is valid?

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- How do we show that an argument form is valid?
  - Construct a truth table. However, this could be tedious.
- We first show the validity of some simple argument forms. These are called *rules of inference*. These may be used to show the validity of more complex argument forms.

Rule of inference	Tautology	Name
$\begin{array}{l} p \\ p \to q \\ \hline \therefore ? \end{array}$	$[p \wedge (p \to q)] \to ?$	Modus ponens
$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore \neg ? \end{array}$	$[\neg q \wedge (p \to q)] \to ?$	Modus tollens
$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore ? \end{array}$	$[(p \to q) \wedge (q \to r)] \to ?$	Hypothetical syllogism
$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore ? \end{array}$	$[(p \vee q) \wedge \neg p] \to ?$	Disjunctive syllogism

Table: Rules of inference.

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$\begin{array}{c} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollens
$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\begin{array}{c} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$	$[(p \vee q) \wedge \neg p] \rightarrow (q)$	Disjunctive syllogism

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$\dfrac{p}{\therefore ?}$	$p \rightarrow ?$	Addition
$\dfrac{p \wedge q}{\therefore ?}$	$(p \wedge q) \rightarrow ?$	Simplification
$\dfrac{\begin{matrix} p \\ q \end{matrix}}{\therefore ?}$	$[(p) \wedge (q)] \rightarrow ?$	Conjunction
$\dfrac{\begin{matrix} p \vee q \\ \neg p \vee r \end{matrix}}{\therefore ?}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow ?$	Resolution

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$\dfrac{p \quad q}{\therefore p \wedge q}$	$[(p) \wedge (q)] \rightarrow (p \wedge q)$	Conjunction
$\dfrac{p \vee q \quad \neg p \vee r}{\therefore q \vee r}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution

Table: Rules of inference.

- Which rule of inference is used in the argument:

  If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have barbecue tomorrow.

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- Show that the hypothesis:
  - "*It is not sunny this afternoon and it is colder than yesterday*."
  - "*We will go swimming only if it is sunny*."
  - "*If we do not go swimming, then we will take a canoe trip*."
  - "*If we take a canoe trip, then we will be home by sunset*."
- lead to the conclusion:
  - "*We will be home by sunset*."

- Show that the following argument is valid. If today is tuesday, I have a test in Mathematics or economics. If my Economics Professor is sick, I will not have a test in Economics. Today is tuesday and my Economics Professor is sick. Therefore I have a test in Mathematics.
- Show that the following argument is valid. If Mohan is a lawyer, then he is ambitious. If Mohan is an early riser, then he does not like idlies. If Mohan is ambitious, then he is an early riser. Then if Mohan is a lawyer, then he does not like idlies.

Resolution Principle

- *Resolution Principle* is another way of showing that an argument is correct.
- Definitions:
  - Literal: A variable or a negation of a variable is called a literal.
  - Sum and Product: A disjunction of literals is called a sum and a conjunction of literals is called a product.
  - Clause: A disjunction of literals is called a clause.
  - Resolvent: For any two clauses C_1 and C_2, if there is a literal L_1 in C_1 that is complementary to literal L_2 in C_2, then delete L_1 and L_2 from C_1 and C_2 respectively and construct the disjunction of the remaining clauses. The constructed clause is a resolvent of C_1 and C_2.
    - $C_1 = P \vee Q \vee R$
    - $C_2 = \neg P \vee \neg S \vee T$
    - What is a resolvent of C_1 and C_2?

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    - $C_1 = P \vee Q \vee R$
    - $C_2 = \neg P \vee \neg S \vee T$
    - What is a resolvent of C_1 and C_2? $Q \vee R \vee \neg S \vee T$

Theorem

Given two clauses C_1 and C_2, a resolvent C of C_1 and C_2 is a logical consequence of C_1 and C_2.

- Example: Modus ponens $(P \wedge (P \rightarrow Q) \rightarrow Q)$
    - C_1: P
    - C_2: $\neg P \vee Q$
    - The resolvent of C_1 and C_2 is Q which is a logical consequence of C_1 and C_2.

Theorem

Given two clauses C_1 and C_2, a resolvent C of C_1 and C_2 is a logical consequence of C_1 and C_2.

Definition (Resolution principle and refutation)

Given a set S of clauses, a (resolution) deduction of C from S is a finite sequence $C_1, ..., C_k$ of clauses such that each C_i either is a clause in S or a resolvent of clauses preceding C and $C_k = C$. A deduction of $\square$ (empty clause) is called a *refutation*.

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Given a set S of clauses, a (resolution) deduction of C from S is a finite sequence $C_1, ..., C_k$ of clauses such that each C_i either is a clause in S or a resolvent of clauses preceding C and $C_k = C$. A deduction of $\square$ (empty clause) is called a *refutation* of a proof of S.

- If there is an argument where $P_1, ..., P_r$ are the premises and C is the conclusion, to get a proof using resolution principle, put $P_1, ..., P_r$ in clause form and add to it $\neg C$ in clause form. From this sequence, if $\square$ can be derived, the argument is valid.

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- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

  - What are the clauses?

- Example:

$$\begin{array}{l} T \to (M \vee E) \\ S \to \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

  - C_1: $\neg T \vee M \vee E$
  - $C_2 : \neg S \vee \neg E$
  - $C_3 : T$
  - $C_4 : S$
  - $C_5 : \neg M$

- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

- C_1: $\neg T \vee M \vee E$
- C_2 : $\neg S \vee \neg E$
- C_3 : T
- C_4 : S
- C_5 : $\neg M$
- C_6 : $\neg T \vee M \vee \neg S$ (resolvent of C_1 and C_2)

- Example:

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  - C_1: $\neg T \vee M \vee E$
  - C_2 : $\neg S \vee \neg E$
  - C_3 : T
  - C_4 : S
  - C_5 : $\neg M$
  - C_6 : $\neg T \vee M \vee \neg S$ (resolvent of C_1 and C_2)
  - C_7 : $M \vee \neg S$ (resolvent of C_3 and C_6)

- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

- C_1: $\neg T \vee M \vee E$
- C_2 : $\neg S \vee \neg E$
- C_3 : T
- C_4 : S
- C_5 : $\neg M$
- C_6 : $\neg T \vee M \vee \neg S$ (resolvent of C_1 and C_2)
- C_7 : $M \vee \neg S$ (resolvent of C_3 and C_6)
- C_8 : M (resolvent of C_4 and C_7)

- Example:

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  - C_1: $\neg T \vee M \vee E$
  - C_2 : $\neg S \vee \neg E$
  - C_3 : T
  - C_4 : S
  - C_5 : $\neg M$
  - C_6 : $\neg T \vee M \vee \neg S$ (resolvent of C_1 and C_2)
  - C_7 : $M \vee \neg S$ (resolvent of C_3 and C_6)
  - C_8 : M (resolvent of C_4 and C_7)
  - C_9 : $\square$ (resolvent of C_5 and C_8)
- Hence, from the resolution principle, the argument is valid.

End