Name: ______________________

ID number: ______________________

There are 3 questions for a total of 10 points.

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1. Answer the following questions:

   (a) (1 point) State true or false: Let $f(n), g(n)$ be functions mapping positive integers to positive real numbers such that $f(n) = O(g(n))$. Then $3^{f(n)} = O(3^{g(n)})$.

   (a) **False**

   (b) (2 points) Give reason for your answer to part (a).

   Solution: Consider $f(n) = 2n$ and $g(n) = n$. For these functions we can show that $f(n) = O(g(n))$ since for all $n \geq 1$, $f(n) \leq 2 \cdot g(n)$. However, $3^{f(n)} = 3^{2n}$ and $3^{g(n)} = 3^n$. For any constant $c > 0$, if $c < 1$, then $3^{f(n)} > c \cdot 3^{g(n)}$ for all $n > 0$. Otherwise, we can show that for all $n \geq \lceil \log_3 c \rceil + 1$, $3^{f(n)} > c \cdot 3^{g(n)}$. This is because if $n \geq \lceil \log_3 c \rceil + 1$, then $3^n > c$, which further implies $3^{2n} > c \cdot 3^n$.

   (c) (1 point) State true or false: Let $f(n), g(n)$ be functions mapping positive integers to positive real numbers such that $f(n) = O(g(n))$. Let $f'(n) = f(5n)$ and $g'(n) = g(5n)$. Then $f'(n) = O(g'(n))$.

   (c) **True**

   (d) (2 points) Give reason for your answer to part (c).

   Solution: Since $f(n)$ is $O(g(n))$, there exists constants c, n_0 such that for all $n \geq n_0, f(n) \leq c \cdot g(n)$. This means that for all $n \geq \lceil n_0/5 \rceil, f(5n) \leq c \cdot g(5n)$. This means that $f(5n)$ is $O(g(5n))$.

2. (1 point) Express the running time of the algorithm below in big-O notation.

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Alg2(A, n)
  - for i = 1 to n
     - for j = 2i to n
        - A[i] ← A[j] + 1
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2. $O(n^2)$

3. Answer the following:

   (a) (1 point) State true or false: For any positive integers m, n and integers a, b, c, d, if $a \equiv b \ (mod\ m)$ and $c \equiv d \ (mod\ n)$, then $ac \equiv bd \ (mod\ mn)$.

   (a) **False**

   (b) (2 points) Give reason for your answer to part (a).

   Solution: We disprove the statement given in part (a) using the following counterexamples: $m = 2, n = 4$ and $a = 1, b = 3, c = 1, d = 5$. So, we have $1 \equiv 3 \ (mod\ 2)$ and $1 \equiv 5 \ (mod\ 4)$ but $1 \not\equiv 15 \ (mod\ 8)$.