COL202: Discrete Mathematical Structures

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Discrete Probability

Theorem (Bayes' Theorem)

Suppose that E and F are events from a sample space S such that $\mathbf{Pr}[E] \neq 0$ and $\mathbf{Pr}[F] \neq 0$. Then

$$\mathbf{Pr}[F|E] = \frac{\mathbf{Pr}[E|F] \cdot \mathbf{Pr}[F]}{\mathbf{Pr}[E|F] \cdot \mathbf{Pr}[F] + \mathbf{Pr}[E|\bar{F}] \cdot \mathbf{Pr}[\bar{F}]}.$$

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- Example: We have two boxes. The first contains two green balls and seven red balls; the second contains four green balls and three red balls. Bob selects a ball by first choosing one of the two boxes at random. He then selects one of the balls in this box at random. If Bob has selected a red ball, what is the probability that he selected a ball from the first box?

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- Example: Suppose that one person in 100,000 has a particular rare disease for which there is a fairly accurate diagnostic test. This test is correct 99.0% of the time when given to a person selected at random who has the disease; it is correct 99.5% of the time when given to a person selected at random who does not have the disease. Given this information can we find
  (a) the probability that a person who tests positive for the disease has the disease?
  (b) the probability that a person who tests negative for the disease does not have the disease?

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- Other Application: Bayesian spam filtering.

Theorem (Generalized Bayes' Theorem)

Suppose that E is an event from a sample space S and that $F_1, ..., F_n$ are mutually exclusive events such that $\cup_{i=1}^{n} F_i = S$. Assume that $\mathbf{Pr}[E] \neq 0$ and $\mathbf{Pr}[F_i] \neq 0$ for $i = 1, 2, ..., n$. Then

$$\mathbf{Pr}[F_j | E] = \frac{\mathbf{Pr}[E|F_j] \cdot \mathbf{Pr}[F_j]}{\sum_{i=1}^{n} \mathbf{Pr}[E|F_i] \cdot \mathbf{Pr}[F_i]}.$$

Definition (Expectation)

The *expected value*, also called the *expectation* or *mean*, of the random variable X on the sample space S is equal to

$$\mathbf{E}[X] = \sum_{s \in S} p(s) \cdot X(s).$$

The *deviation* of X at $s \in S$ is $X(s) - \mathbf{E}[X]$, the difference between the value of X and the mean of X.

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- A fair coin is flipped three times. Let S be the sample space of the eight possible outcomes, and let X be the random variable that assigns to an outcome the number of heads in this outcome. What is the expected value of X?

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Theorem

If X is a random variable and $\mathbf{Pr}[X = r]$ is the probability that $X = r$, so that $\mathbf{Pr}[X = r] = \sum_{s \in S, X(s)=r} p(s)$, then

$$\mathbf{E}[X] = \sum_{r \in X(S)} \mathbf{Pr}[X = r] \cdot r.$$

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- What is the expected value of the sum of the numbers that appear when a pair of fair dice is rolled?

Theorem

The expected number of successes when n mutually independent Bernoulli trials are performed, where p is the probability of success on each trial, is np.

Theorem (Linearity of expectation)

If X_i, $i = 1, 2, ..., n$ with n a positive integer, are random variables on S, and if a and b are real numbers, then
(i) $\mathbf{E}[X_1 + X_2 + ... + X_n] = \mathbf{E}[X_1] + \mathbf{E}[X_2] + ... + \mathbf{E}[X_n]$,
(ii) $\mathbf{E}[aX + b] = a \cdot \mathbf{E}[X] + b$.

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- What is the expected value of the sum of the numbers that appear when a pair of fair dice is rolled?
- What is the expected value of the number of successes when n independent Bernoulli trials are performed, where p is the probability of success on each trial?

- Average-case complexity: Let the sample space I consist of all possible inputs to the algorithm. Let X be a random variable denoting the running time of the algorithm. Then the average-case complexity of the algorithm is $\mathbf{E}[X] = \sum_{i \in I} p(i) \cdot X(i)$.
- What is the average-case complexity of insertion sort if we just count the number of comparisons?

Definition (Geometric distribution)

A random variable X has a *geometric distribution with parameter* p if $\mathbf{Pr}[X = k] = (1-p)^{k-1}p$ for $k = 1, 2, 3, ...,$ where p is a real number with $0 \leq p \leq 1$.

- Example: Suppose that the probability that a coin comes up tails is p. This coin is flipped repeatedly until it comes up tails. What is the expected number of flips until this coin comes up tails?

Theorem

If the random variable X has the geometric distribution with parameter p, then $\mathbf{E}[X] = 1/p$.

End