COL202: Discrete Mathematical Structures

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Counting

Theorem (Permutation with indistinguishable objects)

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!\dots n_k!}.$$

- Example: How many different strings can be made by reordering the letters of the word SUCCESS?

Theorem (Distinguishable objects into distinguishable boxes)

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, ..., k$, equals

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  - This is the same as generating permutations of $\{1, 2, ..., n\}$.
- Total ordering on permutations of $\{1, 2, 3, ..., n\}$:
  - $(a_1, a_2, ..., a_n) < (b_1, b_2, ..., b_n)$ iff there is a j such that $a_1 = b_1, a_2, = b_2, ..., a_{j-1} = b_{j-1}$, and $a_j < b_j$.
- Question: What is the next permutation after $(a_1, a_2, ..., a_n)$?

- How do you generate a combination of n distinct objects?

Discrete Probability

- An *experiment* is a procedure that yields one of a given set of possible outcomes.
- The *sample space* of the experiment is the set of possible outcomes.
- An *event* is a subset of the sample space.

Definition (Probability)

If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the probability of E is $\mathbf{Pr}[E] = \frac{|E|}{|S|}$.

- What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

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- What is the probability that the numbers 11, 4, 17, 39, and 23 are drawn in that order from a bin containing 50 balls labeled with the numbers 1, 2, ..., 50 if (a) the ball selected is not returned to the bin before the next ball is selected and (b) the ball selected is returned to the bin before the next ball is selected?

Theorem

Let E be an event in a sample space S. The probability of the event $\bar{E} = S - E$, the complementary event of E, is given by $\mathbf{Pr}[\bar{E}] = 1 - \mathbf{Pr}[E]$.

- A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

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Theorem

Let E_1 and E_2 be events in the sample space S. Then

$$\mathbf{Pr}[E_1 \cup E_2] = \mathbf{Pr}[E_1] + \mathbf{Pr}[E_2] - \mathbf{Pr}[E_1 \cap E_2].$$

- What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

- Probabilistic reasoning: Suppose you have to decide between two events. Then you use the probability of occurrence of these events in your decision-making.
- Example: Monty Hall three-door puzzle.

- While defining probability, we assume that all outcomes of the experiment are equally likely.
- This is restrictive in most cases.

Definition (Probability distribution)

Let S be a sample space of an experiment with a finite or a countable number of outcomes. Let $p : S \rightarrow [0, 1]$ be a function such that $\sum_{s \in S} p(s) = 1$. p is called a *probability distribution* over S.

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- What probabilities should we assign to the outcomes H (heads) and T (tails) when a fair coin is flipped? What probabilities should be assigned to these outcomes when the coin is biased so that heads comes up twice as often as tails?

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Definition

Suppose that S is a set with n elements. The *uniform distribution* assigns the probability $1/n$ to each element of S.

Definition

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Theorem

If $E_1, E_2, \ldots$ is a sequence of pairwise disjoint events in a sample space S, then

$$\mathbf{Pr}\left[\cup_i E_i\right] = \sum_i \mathbf{Pr}[E_i].$$

Definition (Conditional probability)

Let E and F be events with $\mathbf{Pr}[F] > 0$. The *conditional probability* of E given F, denoted by $\mathbf{Pr}[E|F]$, is defined as

$$\mathbf{Pr}[E|F] = \frac{\mathbf{Pr}[E \cap F]}{\mathbf{Pr}[F]}.$$

Definition (Independence)

The events E and F are independent if and only if $\mathbf{Pr}(E \cap F) = \mathbf{Pr}(E) \cdot \mathbf{Pr}(F)$.

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- Suppose E is the event that a randomly generated bit string of length four begins with a 1 and F is the event that this bit string contains an even number of 1s. Are E and F independent, if the 16 bit strings of length four are equally likely?

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- Assume that each of the four ways a family can have two children is equally likely. Are the events E, that a family with two children has two boys, and F, that a family with two children has at least one boy, independent?

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The events E and F are independent if and only if $\mathbf{Pr}(E \cap F) = \mathbf{Pr}(E) \cdot \mathbf{Pr}(F)$.

Definition (Pairwise and mutual independence)

The events $E_1, E_2, ..., E_n$ are pairwise independent if and only if $\mathbf{Pr}(E_i \cap E_j) = \mathbf{Pr}(E_i) \cdot \mathbf{Pr}(E_j)$ for all pairs of integers i and j with $1 \leq i < j \leq n$. These events are mutually independent if $\mathbf{Pr}(E_{i_1} \cap E_{i_2} \cap ... \cap E_{i_m}) = \mathbf{Pr}(E_{i_1}) \cdot \mathbf{Pr}(E_{i_2}) \cdot ... \cdot \mathbf{Pr}(E_{i_m})$ whenever $i_j, j = 1, 2, ..., m$, are integers with $1 \leq i_1 < i_2 < ... < i_m \leq n$ and $m \geq 2$.

- Each performance of an experiment with two possible outcomes is called a *Bernoulli trial*.
- In general, a possible outcome of a Bernoulli trial is called a success or a failure. If p is the probability of a *success* and q is the probability of a *failure*, it follows that $p + q = 1$.
- We are interested in the probability of success in k trials in an experiment that consists of n mutually independent Bernoulli trials.
- Example: A coin is biased so that the probability of heads is $2/3$. What is the probability that exactly four heads come up when the coin is flipped seven times, assuming that the flips are independent?

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Theorem

The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is $C(n, k)p^k q^{n-k}$.

Definition (Random variable)

A random variable is a function from the sample space of an experiment to the set of real numbers. That is, a random variable assigns a real number to each possible outcome.

- Example: Suppose that a coin is flipped three times. Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome. Then $X(t)$ takes on the following values:
  - $X(HHH) = 3$
  - $X(HHT) = X(HTH) = X(THH) = 2$
  - $X(TTH) = X(THT) = X(TTH) = 1$
  - $X(TTT) = 0$

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Definition (Distribution of random variable)

The *distribution* of a random variable X on a sample space S is the set of pairs $(r, \mathbf{Pr}[X = r])$ for all $r \in X(S)$, where $\mathbf{Pr}[X = r]$ is the probability that X takes the value r. (The set of pairs in this distribution is determined by the probabilities $\mathbf{Pr}[X = r]$ for $r \in X(S)$.)

- Example: Suppose that a coin is flipped three times. Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome. What is the distribution of X?

End