COL202: Discrete Mathematical Structures

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Basic Structure: Sets, Functions, Sequences, Sums, and Matrices

Definition (Set)

A *set* is an unordered collection of objects, called *elements* or *members* of a set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

- Examples:
  - $S_1 = \{1, 3, 5, 7, 9\}$
  - $S_3 = \{x | x \text{ is an odd positive integer less than } 10\}$
  - $S_2 = \{1, 2, 3, ..., 99\}$
  - $\mathbb{N} = \{0, 1, 2, 3, ...\}$, the set of natural numbers.
  - $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$, the set of integers.
  - $\mathbb{Z}^+ = \{1, 2, ...\}$, the set of positive integers.
  - $\mathbb{Q} = \{p/q | p \in \mathbb{Z}, q \in \mathbb{Z}, \text{ and } q \neq 0\}$, the set of rational numbers.
  - $\mathbb{R}$, the set of real numbers.
  - $\mathbb{R}^+$, the set of positive real numbers.
  - $\mathbb{C}$, the set of complex numbers.

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- Examples: Intervals (closed and open)
  - $[a, b] = \{x | a \leq x \leq b\}$
  - $[a, b) = \{x | a \leq x < b\}$
  - $(a, b] = \{x | a < x \leq b\}$
  - $(a, b) = \{x | a < x < b\}$

Definition (Equality of Sets)

Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x (x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

- Are the following sets equal?
  - $\{1, 3, 5\}$ and $\{3, 1, 5\}$
  - $\{1, 3, 5\}$ and $\{1, 1, 3, 3, 3, 5, 5\}$
- A set with no elements is called an *empty set* or *null set*. It is denoted by $\varnothing$ or by $\{\}$.
- A set with one element is called a *singleton set*.

- Venn Diagram
  - Used to represents graphically and indicate relationships between sets.
  - The *Universal set* (all objects under consideration) is represented using a rectangle.
  - Geometric figures (typically circle) inside the rectangle are used to represent sets.
  - Dots are used to represent elements.

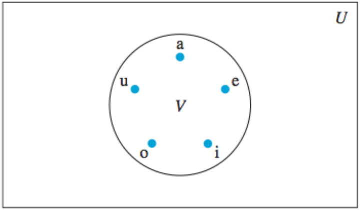

Figure: Venn diagram for the set of vowels

Definition (Subset)

A set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is the subset of the set B.

- For any sets A, B, $A \subseteq B$ iff $\forall x (x \in A \rightarrow x \in B)$ is true.

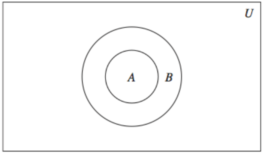

Figure: Venn diagram showing that $A \subseteq B$.

Theorem

For every set S, (i) $\varnothing \subseteq S$ and (ii) $S \subseteq S$.

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- A set A is said to be a *proper subset* of a set B if A is a subset of B but $A \neq B$.
- Write in terms of a quantified expression.

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Theorem

Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$.

Definition (Size of a set)

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

Definition (Infinite set)

A set is said to be infinite if it is not finite. (Example: set of positive integers)

Definition (Power set)

Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

- Examples:
  - $\mathcal{P}(\{1, 2, 3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$
  - $\mathcal{P}(\varnothing) = \{\varnothing\}$.
  - If a set has n elements, how many elements does the power set have?

Definition (Ordered n-tuple)

The ordered n-tuple $(a_1, a_2, ..., a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its n^{th} element.

Definition (Cartesian product of two sets)

Let A and B be sets. The *cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

- Example:
  - $A = \{1, 2\}, B = \{a, b, c\}$
  - $A \times B = ?$
  - $B \times A = ?$

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- Example:
  - $A = \{1, 2\}, B = \{a, b, c\}$
  - $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$
  - $B \times A = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$

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The Cartesian product of the sets $A_1, A_2, ..., A_n$, denoted by $A_1 \times A_2 \times ... \times A_n$, is the set of ordered n-tuples $(a_1, a_2, ..., a_n)$, where a_i belongs to A_i for $i = 1, 2, ..., n$. In other words,

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Definition (Relation)

A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. A relation from a set A to itself is called a relation on A.

- Given a predicate P, and a domain D, we define the *truth set* of P to be the set of elements x in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D | P(x)\}$.
- Examples: Consider predicates $P(x) : |x| = 1$, $Q(x) : x^2 = 2$, and $R(x) : |x| = x$ and let the domain be the set of integers.
  - Truth set of $P(x) =?$
  - Truth set of $Q(x) =?$
  - Truth set of $R(x) =?$

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  - Truth set of $P(x) = \{-1, 1\}$
  - Truth set of $Q(x) = \emptyset$
  - Truth set of $R(x) = \mathbb{N}$

Set operations

Definition (Union of sets)

Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are in A or in B (this includes the element being present in both).

- $A \cup B = \{x | x \in A \vee x \in B\}$.

Definition (Intersection of sets)

Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set that contains those elements that are both in A and in B.

- $A \cap B = \{x | x \in A \wedge x \in B\}$.
- Two sets are called *disjoint* if their intersection is the empty set.
- Show that $|A \cup B| = |A| + |B| - |A \cap B|$.

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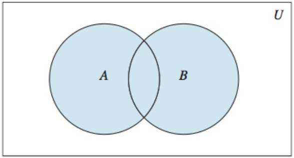
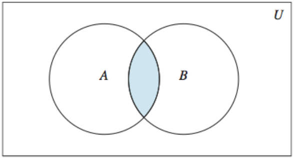


$A \cup B$ is shaded.



$A \cap B$ is shaded.

Definition (Diffrence of sets)

Let A and B be sets. The *difference* of the sets A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement* of B with respect to A.

- $A - B = \{x | x \in A \wedge x \notin B\}$.
- The difference of sets A and B is sometimes denoted by $A \setminus B$.

Definition (Complement of a set)

Let U be the universal set. The *complement* of the set A denoted by $\bar{A}$ is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

- $\bar{A} = \{x \in U | x \notin A\}$.
- Show that $A - B = A \cap \bar{B}$.

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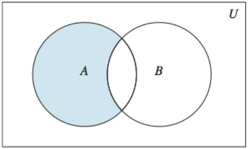
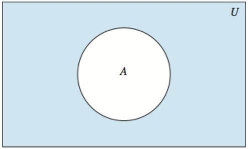


$A - B$ is shaded.



$\bar{A}$ is shaded.

- Show that $\overline{A \cap B} = \bar{A} \cup \bar{B}$.
  - Show (1) $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$, and (2) $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$.
  - Use set builder notation.
  - Use a *membership table*.

Identity	Name
$A \cap U = ?$ $A \cup \varnothing = ?$	Identity laws
$A \cup U = ?$ $A \cap \varnothing = ?$	Domination laws
$A \cup A = ?$ $A \cap A = ?$	Idempotent laws
$\overline{(\overline{A})} = ?$	Complementation law
$A \cup B = B \cup ?$ $A \cap B = B \cap ?$	Commutative laws
$A \cup (B \cup C) = ?$ $A \cap (B \cap C) = ?$	Associative laws
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Table: Set identities.

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Idenitity	**Name**
$\overline{(A \cap B)} =?$ $\overline{(A \cup B)} =?$	De Morgan's laws
$A \cup (A \cap B) =?$ $A \cap (A \cup B) =?$	Absorption laws
$A \cup \overline{A} =?$ $A \cap \overline{A} =?$	Complement laws

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Basic Structures

Set operations

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$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	Complement laws

Table: Set identities.

- Use set identities to show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$.

Basic Structures: Functions

Definition (Function)

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \rightarrow B$.

Definition

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b. The *range*, or image, of f is the set of all images of elements of A. Also,if f is a function from A to B, we say that f maps A to B.

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- Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ assign the square of an integer to this integer.
  - What is the codomain of f?
  - What is the range of f?

Definition (real/integers-valued functions)

A function is called *real-valued* if its codomain is the set of real numbers, and it is called *integer-valued* if its codomain is the set of integers.

Definition (Sum/product of real/integer-valued functions)

Let f_1 and f_2 be functions from A to $\mathbb{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \qquad (f_1 f_2)(x) = f_1(x) f_2(x).$$

- Example: Let f_1 and f_2 be functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. The what are:
  - $(f_1 + f_2)(x) = ?$
  - $(f_1 f_2)(x) = ?$

Definition

Let f be a function from A to B and let S be a subset of A. The image of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t | \exists s \in S(t = f(s))\}.$$

We also use the shorthand $\{f(s) | s \in S\}$ to denote this set.

- Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1$,and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) =?$.

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Definition (One-to-one functions)

A function f is said to be *one-to-one*, or an *injunction*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be *injective* if it is one-to-one.

- Consider a function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $f(x) = x^2$. Is this function one-to-one?

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Definition (Increasing/decreasing functions)

A function f whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$, and *strictly increasing* if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called *decreasing* if $f(x) \geq f(y)$, and *strictly decreasing* if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f. (The word strictly in this definition indicates a strict inequality.)

- Prove or disprove: A strictly increasing function from $\mathbb{R}$ to $\mathbb{R}$ is one-to-one.

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Definition (Onto functions)

A function f from A to B is called *onto*, or a *surjections*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

- Is the function $f(x) = x^2$ from $\mathbb{Z}$ to $\mathbb{Z}$ onto?

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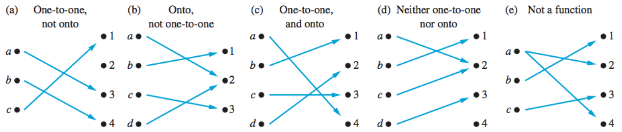
Definition (Bijection)

The function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto. We also say that such a function is *bijective*.



- Suppose that $f : A \rightarrow B$.
  - To show that f is injective: Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.
  - To show that f is not injective: Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
  - To show that is surjective: Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
  - To show that f is not surjective: Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Definition (Inverse function)

Let f be a one-to-one correspondence from the set A to the set B. The *inverse* function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

- Example: Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ with $f(x) = x^2$. Is f invertible?

Definition (Composition of functions)

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.

- Example: Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be functions defined as $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?
- For any function f, what is the composition of f and f^{-1}?
- For any function f, what is the composition of f^{-1} and f?

Definition (Graph of functions)

Let f be a function from the set A to the set B. The *graph* of a function f is the set of ordered pairs $\{(a, b) | a \in A \text{ and } f(a) = b\}$.

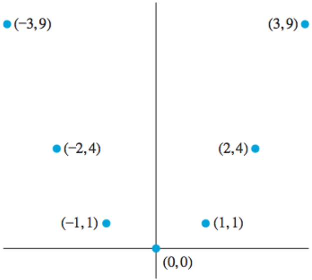


Figure: The graph of $f(x) = x^2$ from $\mathbb{Z}$ to $\mathbb{Z}$.

Definition (Partial functions)

A partial function f from a set A to a set B is an assignment to each element a in a subset of A, called the domain of definition of f, of a unique element b in B. The sets A and B are called the domain and codomain of f, respectively. We say that f is undefined for elements in A that are not in the domain of definition of f. When the domain of definition of f equals A, we say that f is a total function.

- The function $f : \mathbb{Z} \rightarrow \mathbb{R}$ where $f(n) = \sqrt{n}$ is a partial function from $\mathbb{Z}$ to $\mathbb{R}$ where the domain of definition is the set of nonnegative integers.

Sequences and summations

Definition (Sequence)

A sequence is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, ...\}$ or the set $\{1, 2, 3, ...\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

- We use the notation $\{a_n\}$ to describe the sequence.
- Example: $\{a_n\}$ where $a_n = 1/n$. The terms of this sequence, beginning with a_1 is $1, 1/2, 1/3, 1/4, ...$

Definition (Geometric progression)

A geometric progression is a sequence of the form $a, ar, ar^2, ..., ar^n, ...$ where the initial term a and the common ratio r are real numbers.

Definition (Arithmetic progression)

An arithmetic progression is a sequence of the form $a, a+d, a+2d, ..., a+nd, ...$ where the initial term a and the common difference d are real numbers.

Definition (Recurrence relation)

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, ..., a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

- Example: $\{a_n\}$ is a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, ...$ and $a_0 = 3$ and $a_1 = 5$.

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- The *initial conditions* for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect.
- We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a *closed formula*, for the terms of the sequence.

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- Solve the following recurrence relation and the initial condition: $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, ...$ and $a_0 = 2$.

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Cardinality of Sets

Definition

The sets A and B have the same cardinality if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write $|A| = |B|$.

Definition

If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. The cardinality of A is less than the cardinality of B, written as $|A| < |B|$, if there is an injection but no surjection from A to B.

Definition (Countable and uncountable sets)

A set that is either finite or has the same cardinality as the set of positive integers is called *countable*. A set that is not countable is called *uncountable*.

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- Show that the set of odd positive integers is a countable set.

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- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).

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Theorem

Let S be a set. Then $|S| < |\mathcal{P}(S)|$.

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     - Consider any function $f : S \to \mathcal{P}(S)$ and consider the following set defined in terms of this function: $A = \{x | x \notin f(x)\}$
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     - Claim 2.1: There does not exist an element $s \in S$ such that $f(s) = A$.
     - Proof: For the sake of contradiction, assume that there is an $s \in S$ such that $f(s) = A$. The following bi-implications follow:

$$\begin{aligned} s \in A \quad &\leftrightarrow \quad s \in \{x | x \notin f(x)\} \\ &\leftrightarrow \quad s \notin f(s) \\ &\leftrightarrow \quad s \notin A \end{aligned}$$

This is a contradiction. Hence the statement of the claim holds. □

End