COL202: Discrete Mathematical Structures

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Definition (Logical equivalence)

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

- Are these logically equivalent:
  - $\neg \forall x P(x)$ and $\exists x \neg P(x)$?
  - $\neg \exists x P(x)$ and $\forall x \neg P(x)$?

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- These are logically equivalent:
  - $\neg \forall x P(x)$ and $\exists x \neg P(x)$
  - $\neg \exists x P(x)$ and $\forall x \neg P(x)$
- These rules for negation of quantifiers are called *De Morgan's laws for quantifiers*.
- Show that $\neg \forall x (P(x) \rightarrow Q(x))$ and $\exists x (P(x) \wedge \neg Q(x))$ are logically equivalent.

- Analyze complex natural language sentences.
  - Example: "*Every student in this class has visited either Delhi or Mumbai.*"
- Translate system specifications.
  - Example: "*Every mail message larger than one megabyte will be compressed.*"
- Deriving conclusions from statements:
  - Examples: Consider the following statements
    - "*All lions are fierce.*"
    - "*Some lions do not drink coffee.*"
  - From the above two sentences can we make the following conclusion?
    - "*Some fierce creatures do not drink coffee.*"

- Nested Quantifiers: Two quantifiers are nested if one is within the scope of the other.
  - Example:
    - $\forall x \exists y (x + y = 0)$.
    - We may write the above as $\forall x Q(x)$, where $Q(x) = \exists y (x + y = 0)$.
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- The order of the quantifiers is important. Consider the following examples:
  - Let $Q(x, y)$ denote $(x + y = 0)$ and let the domain for x, y consist of all real numbers.
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Statement	When True	When False
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	?	?
$\forall x \exists y P(x, y)$	?	?
$\exists x \forall y P(x, y)$	?	?
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	?	?

Table: Nested quantification of two variables.

Statement	When True	When False
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is True for every pair x, y	There is a pair x, y for which $P(x, y)$ is False
$\forall x \exists y P(x, y)$	For every x there is a y such that $P(x, y)$ is True	There is an x such that $P(x, y)$ is False for every y
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is True for every y	For every x there is a y for which $P(x, y)$ is False.
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$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is True for every y	For every x there is a y for which $P(x, y)$ is False.
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- Determine the truth value of these statements when the domain of all variables consists of all real numbers and $Q(x, y, z)$ denotes $x + y = z$.
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  - $\forall x \forall y \exists z Q(x, y, z)$? True
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- Translating mathematical statements into logical expressions involving nested quantifiers.
  - "*The sum of two positive integers is always positive.*"
  - "*Every real number except 0 has a multiplicative inverse.*"

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- Translating mathematical statements into logical expressions involving nested quantifiers.
  - "*The sum of two positive integers is always positive.*"
    - $\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$
  - "*Every real number except 0 has a multiplicative inverse.*"
    - $\forall x((x \neq 0) \rightarrow \exists y(xy = 1))$

- Translating nested quantifiers into English.
  - Example:
    - $F(a,b)$: "*a and b are friends.*"
    - The domain consists of all students in the Institute.
    - Translate: $\exists x \forall y \forall z (F(x,y) \land F(x,z) \land (y \neq z) \rightarrow \neg(F(y,z)))$

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        - "*There is a student none of whose friends are also friends with each other.*"

- Translating English sentences into logical expressions:
- Consider following examples:
  - Translate: "*If a person is female and is a parent, then this person is someone's mother.*"
  - Translate: "*Everyone has exactly one best friend.*"
  - Translate: "*There does not exist a woman who has taken a flight on every airline in the world.*"

Warm-up exercise

- You must be familiar with the definition of Big-O from Data Structures. Note that the definition is an interesting example of nested quantifiers.

Definition (Big-O)

Let $f(n)$ and $g(n)$ denote functions mapping positive integers to positive real numbers. The function $f(n)$ is said to be $O(g(n))$ (or $f(n) = O(g(n))$) in short if and only if there exists constants $C, n_0 > 0$ such that for all $n \geq n_0, f(n) \leq C \cdot g(n)$.

- How would you argue that $5n^2 + 3n + 1$ is $0(n^2)$?
- How would you argue that 2^n is not $O(n)$?

Rules of Inference

- Consider the following argument.
  - "*If you have a current password, then you can log onto the network*."
  - "*You have a current password*."
  - Therefore "*you can log onto the network*."
- Is this a valid argument? Why?

- Argument: A sequence of statements that end with a conclusion.
- Valid argument: The conclusion must follow from the truth of the preceding statements (known as *premises*). An argument is valid if and only if it is impossible for all the premises to be true and conclusion to be false.
- Rules of inference: Templates for obtaining new statements from already available statements or in other words templates for constructing valid arguments.

- Consider the following argument.
  - "*If you have a current password, then you can log onto the network*."
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  - "*If you have a current password, then you can log onto the network.*"
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- Is this a valid argument? Why?
- Let us write this argument more concisely.
  - Propositions:
    - p: "*You have a current password.*"
    - q: "*You can log onto the network.*"
  - Argument:

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

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    - p: "*You have a current password*."
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$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

- We know that for any propositions p, q, r, $((p \to q) \wedge p) \to q$ is a tautology.
- So, the initial argument is a valid argument.

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- We know that for any propositions p, q, r, $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology.
- So, the initial argument is a valid argument.
- Moreover, if we plug in any propositions into p and q such that the premises $p \rightarrow q$ and p are true for these propositions, then concluding q from these premises is a valid argument.
- This is called an *argument form*. The validity of an argument follows from the validity of the argument form.

Definition (Argument and argument form)

An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*. An argument is valid if the truth of all its premises implies that the conclusion is true.

An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* if no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

- How do we show that an argument form is valid?

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An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*. An argument is valid if the truth of all its premises implies that the conclusion is true.
An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* if no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

- How do we show that an argument form is valid?
  - Construct a truth table. However, this could be tedious.
- We first show the validity of some simple argument forms. These are called *rules of inference*. These may be used to show the validity of more complex argument forms.

Rule of inference	Tautology	Name
p $p \rightarrow q$ ——— $\therefore ?$	$[p \wedge (p \rightarrow q)] \rightarrow ?$	Modus ponens
$\neg q$ $p \rightarrow q$ ——— $\therefore \neg ?$	$[\neg q \wedge (p \rightarrow q)] \rightarrow ?$	Modus tollens
$p \rightarrow q$ $q \rightarrow r$ ——— $\therefore ?$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow ?$	Hypothetical syllogism
$p \vee q$ $\neg p$ ——— $\therefore ?$	$[(p \vee q) \wedge \neg p] \rightarrow ?$	Disjunctive syllogism

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$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollens
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \rightarrow (q)$	Disjunctive syllogism

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$\dfrac{p}{\therefore ?}$	$p \rightarrow ?$	Addition
$\dfrac{p \wedge q}{\therefore ?}$	$(p \wedge q) \rightarrow ?$	Simplification
$\dfrac{p \quad q}{\therefore ?}$	$[(p) \wedge (q)] \rightarrow ?$	Conjunction
$\dfrac{p \vee q \quad \neg p \vee r}{\therefore ?}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow ?$	Resolution

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$\dfrac{p \vee q \quad \neg p \vee r}{\therefore q \vee r}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution

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End