COL202: Discrete Mathematical Structures

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Administrative Information

- Textbook: Discrete Mathematics and its Applications by *Kenneth H. Rosen*.
- Gradescope: A paperless grading system. Use the course code **M4V24Z** to register in the course on Gradescope. Use only your IIT Delhi email address to register on Gradescope.
- Course webpage: `http://www.cse.iitd.ac.in/~rjaiswal/Teaching/2018/COL202`.
  - The site will contain course information, references, homework/tutorial problems. Please check this page regularly.

- Simplify complex sentences and enable to logically analyze them.
- Translate system specification expressed in natural language into unambiguous logical expressions.
  - Example:
    - "The diagnostic message is stored in the buffer or is retransmitted."
    - "The diagnostic message is not stored in the buffer."
    - "If the diagnostic message is stored in the buffer, then it is retransmitted."
    - "The diagnostic message is not retransmitted."
  - <u>Consistency</u>: Whether all the specifications can be satisfied simultaneously.
  - Are these specifications consistent?

- Simplify complex sentences and enable to logically analyze them.
- Translate system specification expressed in natural language into unambiguous logical expressions.
- Resolve complex puzzling scenarios.
  - Example:
    - An island has two kinds of inhabitants, knights and knaves. Knights always tell the truth and Knaves always lie. You meet two people on this island A and B. What are A and B if A says "B is a knight" and B says "The two of us are opposite types"?

Definition (Tautology and Contradiction)

A compound proposition that is always true, no matter what the truth values of the proposition that occurs in it, is called a tautology. A compound proposition that is always false is called a contradiction. A compound proposition that is neither a tautology nor a contradiction is called a contingency.

- Examples:
  - $(p \vee \neg p)$ is a tautology.
  - $(p \wedge \neg p)$ is a contradiction.

Definition (Logical equivalence)

A compound proposition p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

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- Show that p and q are logically equivalent if and only if the columns giving their truth values match.
- Show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

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- Show that $p \rightarrow q \equiv \neg p \vee q$.

Equivalence	Name
$p \wedge T \equiv ?$ $p \vee F \equiv ?$	Identity laws
$p \vee T \equiv ?$ $p \wedge F \equiv ?$	Domination laws
$p \vee p \equiv ?$ $p \wedge p \equiv ?$	Idempotent laws
$\neg(\neg p) \equiv ?$	Double negation law
$p \vee q \equiv ?$ $p \wedge q \equiv ?$	Commutative laws
$(p \vee q) \vee r \equiv ?$ $(p \wedge q) \wedge r \equiv ?$	Associative laws
$p \vee (q \wedge r) \equiv ?$ $p \wedge (q \vee r) \equiv ?$	Distributive laws

Table: Logical equivalences.

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$p \vee (p \wedge q) \equiv ?$ $p \wedge (p \vee q) \equiv ?$	Absorption laws
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- Show that $\neg(p \rightarrow q) \equiv (p \wedge \neg q)$.
- Show that $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$.
- Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

- Consider the following two statements:
  - *"All computers connected to the Institute network are functioning properly."*
  - *"Computer-1 is connected to the Institute network."*
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- Suppose there are only two computers in the institute. Consider the following propositions:
  - p: Computer-1 is connected to the network.
  - q: Computer-2 is connected to the network.
  - r: Computer-1 is functioning properly.
  - s: Computer-2 is functioning properly.
- We can write $(p \rightarrow r) \wedge (q \rightarrow s) \wedge p$.

Logic

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- Suppose there are only two computers on the institute. Consider the following propositions:
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- We can write $(p \rightarrow r) \wedge (q \rightarrow s) \wedge p$.
- Now, suppose there are 10,000 computers in the institute?

Predicate Logic

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- Is it ok to make the conclusion that Computer-1 is functioning properly?
- Suppose there are 10,000 computers in the institute?
- Consider the following concise way of writing propositions:
  - $P(x)$: x is connected to the institute network.
    - x can take values Computer-1, Computer-2 etc.
    - P denotes the *predicate* "is connected to the institute network."
    - $P(x)$ can be thought of the value of the *propositional function* P at x.

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- Consider the following concise way of writing propositions:
  - $P(x)$: x is connected to the institute network.
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- What we would like to say is that for any assignment of x from the set $\{$Computer-1, ..., Computer-10000$\}$, $P(x) \rightarrow R(x)$.

- Quantification expresses the extent to which a predicate is true over a range of elements.
- There are two types of quantification:
  - *Universal quantification* which tells that a predicate is true for every element under consideration.
  - *Existential quantification* tells us that there is one or more element under consideration for which the predicate is true.
- The area of logic that deals with predicates and quantifiers is called *predicate calculus*.

Definition (Universal quantification)

The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the *domain*." The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the *universal quantifier*. We read $\forall x P(x)$ as "for all x $P(x)$." An element for which $P(x)$ is false is called a *counterexample* of $\forall x P(x)$.

- Examples:
  - Let $P(x) : x + 1 > x$. The truth value of the quantification $\forall x P(x)$ is true when the domain consists of all real numbers.

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Definition (Existential quantification)

The existential quantification of $P(x)$ is the statement "there exists an element x in the domain such that $P(x)$." We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

- Examples:
  - Let $P(x) : x^2 \leq 0$. What is the truth value of $\exists x P(x)$ when the domain consists of all integers?

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    - $\forall x < 0 (x^2 > 0)$
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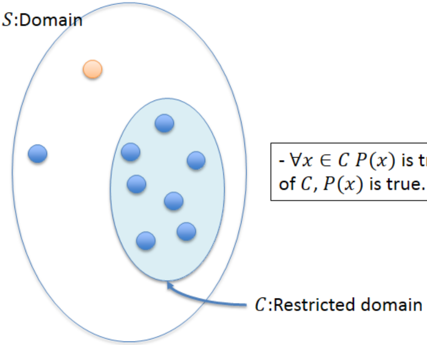

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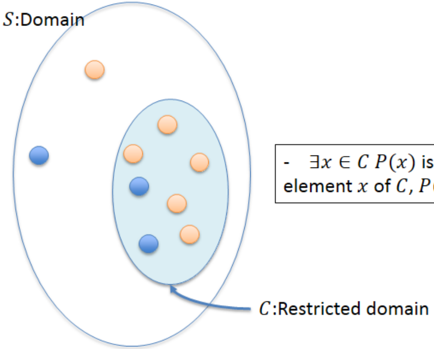

- $\exists x \in C\ P(x)$ is true when for at least one element x of C, $P(x)$ is true.

- Quantifiers with restricted domain:
  - What does the following mean when the domain consists of all real numbers:
    - $\forall x < 0 (x^2 > 0)$: $\forall x (x < 0 \rightarrow x^2 > 0)$
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- More definitions: Binding and free variables, scope.
  - Binding variable: When a quantifier is used on a variable x, we say that this occurence of the variable is *bound*.
  - Free variable: An occurence of a variable that is not bound by a quantifier or set equal to a particular value is said to be *free*.
  - Scope of quantifier: The part of a logical expression to which a quantifier is applied is called the *scope* of this quantifier.

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  - Examples:
    - $\exists x(x + y = 1)$
    - $\forall x(P(x) \wedge Q(x)) \vee \forall x R(x)$

Definition (Logical equivalence)

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

- Are these logically equivalent:
  - $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$?
  - $\exists x(P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$?
  - $\forall x(P(x) \vee Q(x))$ and $\forall x P(x) \vee \forall x Q(x)$?
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- These are logically equivalent:
  - $\neg \forall x P(x)$ and $\exists x \neg P(x)$
  - $\neg \exists x P(x)$ and $\forall x \neg P(x)$
- These rules for negation of quantifiers are called *De Morgan's laws for quantifiers*.
- Show that $\neg \forall x (P(x) \rightarrow Q(x))$ and $\exists x (P(x) \land \neg Q(x))$ are logically equivalent.

End