Name: ______________________

Entry number: ______________________

There are 2 questions for a total of 10 points.

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1. Answer the following questions.

   (a) (1 point) State true or false (no reasons required): The probability of having an empty bin when throwing k distinguishable balls randomly into n distinguishable bins is the same as the probability of having an empty bin when throwing k indistinguishable balls randomly into n distinguishable bins.

   (a) **True**

   This is for explanation. You were not expected to write this.
   Note that the randomness is over the random choices made when throwing the balls. Suppose there is an n-sided dice that is rolled every time the ball is supposed to thrown randomly. The sample space is $\{1, ..., n\} \times \{1, ..., n\} \times ... \times \{1, ..., n\}$ (all possible dice rolls) and the number of favourable cases is the same in both the scenarios.

   (b) (4 points) Suppose you flip a biased coin that turns heads with probability p. What is the probability that you get even number of heads in n coin tosses. You have to give a concise expression.

   (b) $\frac{1}{2} + \frac{1}{2} \cdot (1 - 2p)^n$

   Solution: Let Q denote this probability. Then we have

$$Q = \binom{n}{0} \cdot p^0 \cdot (1-p)^n + \binom{n}{2} \cdot p^2 \cdot (1-p)^{n-2} + \binom{n}{4} \cdot p^4 \cdot (1-p)^{n-4} + ...$$

   We know that

$$((1-p)+(p))^n = \binom{n}{0} \cdot p^0 \cdot (1-p)^n + \binom{n}{1} \cdot p^1 \cdot (1-p)^{n-1} + \binom{n}{2} \cdot p^2 \cdot (1-p)^{n-2} + \binom{n}{3} \cdot p^3 \cdot (1-p)^{n-3} + ...$$

   and

$$((1-p)+(-p))^n = \binom{n}{0} \cdot p^0 \cdot (1-p)^n - \binom{n}{1} \cdot p^1 \cdot (1-p)^{n-1} + \binom{n}{2} \cdot p^2 \cdot (1-p)^{n-2} - \binom{n}{3} \cdot p^3 \cdot (1-p)^{n-3} + ...$$

   Adding the last two equations, we get $1 + (1-2p)^n = 2 \cdot Q$. This gives $Q = \frac{1}{2} + \frac{1}{2} \cdot (1-2p)^n$.

2. (5 points) A fair coin is tossed repeatedly until two consecutive heads are tossed. Find the probability that the coin was tossed 11 times. Show calculations in the space below.

2. $\frac{55}{2048}$

Solution: The total number of outcomes of n fair coin tosses are 2^n. We will count the total number of favourable cases. If $n \geq 3$, then any favourable case, can be denoted by a string in $\{H, T\}^n$ that ends with THH and does not have consecutive H's at any previous position. Let a_m denote the number of H/T strings of length m that does not have consecutive H's. Then we have the following recurrence relation $\forall m \geq 2, a_m = a_{m-1} + a_{m-2}$. Also, $a_1 = 2$ and $a_2 = 3$. The recurrence relation is true since strings of length m not having consecutive H's is made of strings of length $m-1$ that do not have consecutive H's followed by a T and strings of length $m-2$ that do not have consecutive H's followed by TH. Note that the solution of the recurrence relation is $\forall n \geq 1, a_n = F_{n+2}$, where F_k denotes the k^{th} Fibonacci number ($F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34, F_{10} = 55, F_{11} = 89, F_{12} = 144, ...$).

Note that the number of favourable coin tosses $\forall n \geq 4$ is given by $a_{n-3} = F_{n-1}$. Let Q_n denote the probability the coin was tossed n times. Then we have $Q_1 = 0$, $Q_2 = 1/4$, $Q_3 = 1/8$, and $\forall n \geq 4, Q_n = \frac{F_{n-1}}{2^n}$. In general, we can write $\forall n \geq 1, Q_n = \frac{F_{n-1}}{2^n}$. So, $Q_{11} = \frac{F_{10}}{2^{11}} = \frac{55}{2048}$.