# COL866: Foundations of Data Science

Ragesh Jaiswal, IITD

## Spectral Graph Theory:
## Eigenvalues and graph properties

- We shall work with $d$-regular undirected graphs.
- It will be convenient to work with the matrix $L = I - \frac{1}{d}A$ instead of the adjacency matrix $A$.
- The matrix $L$ defined above is called the Normalized Laplacian Matrix of the graph.
- We prove the following basic results of spectral graph theory.

**Theorem**

*Let $G$ be a $d$-regular undirected graph, and $L = I - \frac{1}{d}A$ be its normalized Laplacian matrix. Let $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be the real eigenvalues of $L$ with multiplicities. Then*

1. $\lambda_1 = 0$ *and* $\lambda_n \leq 2$.
2. $\lambda_k = 0$ *if and only if* $G$ *has at least* $k$ *connected components.*
3. $\lambda_n = 2$ *if and only if at least one of the connected components of* $G$ *is bipartite.*

- Given a $d$-regular undirected graph with normalised graph laplacian $L = I - \frac{1}{d}A$ having eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq 2$.
- We know that the second eigenvalue $\lambda_2 = 0$ if and only if $G$ has at least two connected components.
- In other words, the second eigenvalue $\lambda_2 = 0$ if and only if $\phi(G) = 0$.
- We will prove an *approximate* version of this result that says that $\lambda_2$ is small if and only if $\phi(G)$ is small.

**Theorem (Cheeger's Inequality)**

$\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2 \cdot \lambda_2}$.

**Theorem (Cheeger's Inequality)**

$\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2 \cdot \lambda_2}$.

- First we will prove the following direction.

**Lemma**

$\lambda_2 \leq \sigma(G) \leq 2\phi(G)$.

## Lemma

$\lambda_2 \leq \sigma(G) \leq 2\phi(G)$.

## Proof sketch

- We can write:

$$\sigma(G) = \min_{\mathbf{x}\in\{0,1\}^n - \{\mathbf{0},\mathbf{1}\}} \frac{\sum_{\{u,v\}\in E} |\mathbf{x}_u - \mathbf{x}_v|}{\frac{d}{n}\sum_{\{u,v\}} |\mathbf{x}_u - \mathbf{x}_v|}$$

$$= \min_{\mathbf{x}\in\{0,1\}^n - \{\mathbf{0},\mathbf{1}\}} \frac{\sum_{\{u,v\}\in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{\frac{d}{n}\sum_{\{u,v\}} (\mathbf{x}_u - \mathbf{x}_v)^2}$$

**Lemma**

$\lambda_2 \leq \sigma(G) \leq 2\phi(G)$.

**Proof sketch**

- We can write:

$$\sigma(G) = \min_{x \in \{0,1\}^n - \{\mathbf{0},\mathbf{1}\}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{\frac{d}{n} \sum_{\{u,v\}} (\mathbf{x}_u - \mathbf{x}_v)^2}$$

- Also, we have

$$\lambda_2 = \min_{x \in \mathbb{R}^n - \{\mathbf{0}\}, \mathbf{x} \perp \mathbf{1}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{d \cdot \sum_v \mathbf{x}_v^2}$$

### Lemma

$\lambda_2 \leq \sigma(G) \leq 2\phi(G)$.

### Proof sketch

- We can write:

$$\sigma(G) = \min_{x \in \{0,1\}^n - \{\mathbf{0},\mathbf{1}\}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{\frac{d}{n} \sum_{\{u,v\}} (\mathbf{x}_u - \mathbf{x}_v)^2}$$

- Also, we have

$$\lambda_2 = \min_{x \in \mathbb{R}^n - \{\mathbf{0}\}, \mathbf{x} \perp \mathbf{1}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{d \cdot \sum_v \mathbf{x}_v^2}$$

$$\stackrel{?}{=} \min_{x \in \mathbb{R}^n - \{\mathbf{0}\}, \mathbf{x} \perp \mathbf{1}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{\frac{d}{n} \cdot \sum_{\{u,v\}} (\mathbf{x}_u - \mathbf{x}_v)^2}$$

## Lemma

$\lambda_2 \leq \sigma(G) \leq 2\phi(G)$.

## Proof sketch

- We can write:

$$\sigma(G) = \min_{x \in \{0,1\}^n - \{\mathbf{0},\mathbf{1}\}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{\frac{d}{n} \sum_{\{u,v\}} (\mathbf{x}_u - \mathbf{x}_v)^2}$$

- Also, we have

$$\lambda_2 = \min_{x \in \mathbb{R}^n - \{\mathbf{0}\}, \mathbf{x} \perp \mathbf{1}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{d \cdot \sum_v \mathbf{x}_v^2}$$

$$= \min_{x \in \mathbb{R}^n - \{\mathbf{0}\}, \mathbf{x} \perp \mathbf{1}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{\frac{d}{n} \cdot \sum_{\{u,v\}} (\mathbf{x}_u - \mathbf{x}_v)^2}$$

$$\stackrel{?}{=} \min_{x \in \mathbb{R}^n - \{\mathbf{0},\mathbf{1}\}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{\frac{d}{n} \cdot \sum_{\{u,v\}} (\mathbf{x}_u - \mathbf{x}_v)^2}$$

**Lemma**

$\lambda_2 \leq \sigma(G) \leq 2\phi(G)$.

**Proof sketch**

- We can write:

$$\sigma(G) = \min_{x \in \{0,1\}^n - \{\mathbf{0},\mathbf{1}\}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{\frac{d}{n} \sum_{\{u,v\}} (\mathbf{x}_u - \mathbf{x}_v)^2}$$

- Also, we have

$$\begin{aligned} \lambda_2 &= \min_{x \in \mathbb{R}^n - \{\mathbf{0}\}, \mathbf{x} \perp \mathbf{1}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{d \cdot \sum_v \mathbf{x}_v^2} \\ &= \min_{x \in \mathbb{R}^n - \{\mathbf{0}\}, \mathbf{x} \perp \mathbf{1}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{\frac{d}{n} \cdot \sum_{\{u,v\}} (\mathbf{x}_u - \mathbf{x}_v)^2} \\ &= \min_{x \in \mathbb{R}^n - \{\mathbf{0},\mathbf{1}\}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}_u - \mathbf{x}_v)^2}{\frac{d}{n} \cdot \sum_{\{u,v\}} (\mathbf{x}_u - \mathbf{x}_v)^2} \end{aligned}$$

- So, $\lambda_2 \leq \sigma(G)$.

□

End