COL866: Foundations of Data Science

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Spectral Graph Theory: Eigenvalues and graph properties

Spectral Graph Theory

- Graphs are used to model pairwise relationship between objects. The objects are denoted by vertices and pairwise relationships are denoted by edges.
- Classical techniques have had a lot of success in solving various problems on graphs such as shortest path, minimum spanning tree etc.
- More recently, algebraic techniques have been successful for certain graph problems. Such techniques involve analysing the adjacency matrix of the graph.
- The relationship between the structural properties of a given graph and the algebraic properties of the underlying (appropriate variant of) adjacency matrix is key in such analysis.

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- More recently, algebraic techniques have been successful for certain graph problems. Such techniques involve analysing the adjacency matrix of the graph.
- The relationship between the structural properties of a given graph and the algebraic properties of the underlying (appropriate variant of) adjacency matrix is key in such analysis.
- Spectral Graph Theory studies how eigenvalues of the adjacency matrix of a graph (an algebraic quantity) relates to combinatorial properties of the graph.

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- Consider a communication network setting. Adding edges has an associated cost. So, ideally we would like to have good connectivity with small number of total edges.
- What does it mean to have good connectivity?
- An intuitive notion is that removing a small fraction of edges of the graph should not disconnect large portions of the graph.
- Sparsity and edge expansion discussed next, are useful in formalizing the notion of connectivity.

- Given a graph $G = (V, E)$, a cut in the graph is a partition of vertices into sets S and $V - S$. It is represented as a tuple $(S, V - S)$.

Definition (Sparsity)

The sparsity of a cut $(S, V - S)$, denoted by $\sigma(S)$, is defined as $\sigma(S) = \frac{|V|^2}{|E|} \cdot \frac{|E(S, V-S)|}{|S| \cdot |V-S|}$.

- In other words, the sparsity of the cut $(S, V - S)$ is the fraction of edges across the cut divided by the fraction of vertex pairs across the partition (i.e., the fraction of edges in the ideal case of a complete graph).

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- Note that if the graph is d-regular, then $\sigma(S) = \frac{|E(S,V-S)|}{\frac{d}{|V|} \cdot |S| \cdot |V-S|}$
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- Claim 2: $\frac{1}{2}\sigma(G) \leq \phi(G) \leq \sigma(G)$.
- Constant degree graphs with constant expansion are sparse graphs with good connectivity property.

Theorem

Let $G = (V, E)$ be a regular graph of expansion ϕ. Then, after an $\varepsilon < \phi$ fraction of edges are adversarially removed, the graph has a connected component that spans at least $(1 - \frac{\varepsilon}{2\phi})$ fraction of vertices.

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- In a d-regular expander (i.e., constant expansion), the removal of k edges can cause $O(k/d)$ vertices to be disconnected from the remaining giant component.
- It is always possible to disconnect k/d vertices after removing k edges. So, the reliability of an expander is the best possible.

Linear Algebra Recap: Eigenvalues

- The *conjugate* of a complex number $x = a + ib$ is denoted by $\bar{x}$ and has value $\bar{x} = a - ib$.
- For a complex matrix $M \in \mathbb{C}^{m \times n}$, the conjugate transpose of M, denoted by $M^\star \in \mathbb{C}^{n \times m}$ such that $(M^\star)_{i,j} = \bar{M}_{j,i}$.
- The inner product of two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\star \mathbf{y}$.

Definition (Eigenvalue and eigenvector)

For a square matrix $M \in \mathbb{C}^{n \times n}$, $\lambda \in \mathbb{C}$ is a scalar, $\mathbf{x} \in \mathbb{C}^n - \{\mathbf{0}\}$ is a non-zero vector and

$$M\mathbf{x} = \lambda \mathbf{x}$$

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- The equation $M\mathbf{x} = \lambda \mathbf{x}$ can alternatively be written as $(M - \lambda I)\mathbf{x} = \mathbf{0}$.
- This is equivalent to $det(M - \lambda I) = 0$.
- The LHS of the equation is a polynomial of degree n in λ and so has n solutions (counting multiplicities).
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- We will be studying the eigenvalues of adjacency matrices of undirected graphs which are real and symmetric. Next, we look at some properties of such matrices.
- A square matrix $M \in \mathbb{C}^{n \times n}$ is called Hermitian iff $M = M^{\star}$.

Lemma

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Let $M \in \mathbb{R}^{n \times n}$ be a real symmetric matrix and $\lambda_1 \leq ... \leq \lambda_n$ be its real eigenvalues (counted with multiplicities) sorted in non-decreasing order. Let $\mathbf{x}_1, ..., \mathbf{x}_k$, $k < n$, be orthonormal vectors such that $M\mathbf{x}_i = \lambda_i \mathbf{x}_i$ for $i = 1, ..., k$. Then

$$\lambda_{k+1} = \min_{\mathbf{x} \in \mathbb{R}^n - \{\mathbf{0}\} : \mathbf{x} \perp \mathbf{x}_1 ..., \mathbf{x}_k} \frac{\mathbf{x}^T M \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$

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- We need the following lemma.

Lemma

Let $M \in \mathbb{R}^{n \times n}$ be a real symmetric matrix, and let $\mathbf{x}_1, ..., \mathbf{x}_k, k < n$ be orthogonal eigenvectors of M. Then there is an eigenvector $\mathbf{x}_{k+1}$ of M that is orthogonal to $\mathbf{x}_1, ..., \mathbf{x}_k$.

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- The above lemma has the following interesting corollary.

Corollary (Spectral theorem)

Let $M \in \mathbb{R}^{n \times n}$ be a real symmetric matrix, and $\lambda_1, ..., \lambda_n$ be its real eigenvalues, with multiplicities; then there are orthonormal vectors $\mathbf{x}_1, ..., \mathbf{x}_n, \mathbf{x}_i \in \mathbb{R}^n$ such that $\mathbf{x}_i$ is an eigenvector of λ_i.

Theorem

Let $M \in \mathbb{R}^{n \times n}$ be a real symmetric matrix and $\lambda_1 \leq ... \leq \lambda_n$ be its real eigenvalues (counted with multiplicities) sorted in non-decreasing order. Let $\mathbf{x}_1, ..., \mathbf{x}_k$, $k < n$, be orthonormal vectors such that $M\mathbf{x}_i = \lambda_i \mathbf{x}_i$ for $i = 1, ..., k$. Then

$$\lambda_{k+1} = \min_{\mathbf{x} \in \mathbb{R}^n - \{\mathbf{0}\} : \mathbf{x} \perp \mathbf{x}_1 ..., \mathbf{x}_k} \frac{\mathbf{x}^T M \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$

and any minimizer is an eigenvector of λ_{k+1}.

Proof sketch

- Applying the lemma repeatedly, we can obtain $n - k$ orthogonal vectors $\mathbf{x}_{k+1}, ..., \mathbf{x}_n$, where $\mathbf{x}_i$ is an eigenvector of λ_i.
- Claim 1: $\mathbf{x}_{k+1}$ is a feasible solution for the minimisation problem and has cost λ_{k+1}. So, the minimum us at most λ_{k+1}.
- Claim 2: For any arbitrary feasible solution $\mathbf{x}$, we can write $\mathbf{x} = \sum_{i=1k+1}^{n} a_i \mathbf{x}_i$ and this has cost $\frac{\sum_{i=k+1}^{n} \lambda_i a_i^2}{\sum_{i=k+1}^{n} a_i^2} \geq \lambda_{k+1}$.
- Claim 3: Any minimiser $\mathbf{x}$ is a linear combination of eigenvectors of λ_{k+1} and hence is an eigenvector of λ_{k+1}.

Theorem

Let $M \in \mathbb{R}^{n \times n}$ be a real symmetric matrix and $\lambda_1 \leq ... \leq \lambda_n$ be its real eigenvalues (counted with multiplicities) sorted in non-decreasing order. Let $\mathbf{x}_1, ..., \mathbf{x}_k$, $k < n$, be orthonormal vectors such that $M\mathbf{x}_i = \lambda_i \mathbf{x}_i$ for $i = 1, ..., k$. Then

$$\lambda_{k+1} = \min_{\mathbf{x} \in \mathbb{R}^n - \{\mathbf{0}\} : \mathbf{x} \perp \mathbf{x}_1 ..., \mathbf{x}_k} \frac{\mathbf{x}^T M \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$

and any minimizer is an eigenvector of λ_{k+1}.

- The above theorem may also be written in the following manner:

Corollary

Let $M \in \mathbb{R}^{n \times n}$ be a real symmetric matrix, and $\lambda_1 \leq ... \leq \lambda_n$ its eigenvalues, counted with multiplicities and sorted in non-decreasing order. Then

$$\lambda_k = \min_{k-dimensional\ subspace\ V\ of\ \mathbb{R}^n} \left\{ \max_{\mathbf{x} \in V - \{\mathbf{0}\}} \frac{\mathbf{x}^T M \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \right\}$$

The basics of spectral graph theory

- We shall work with d-regular undirected graphs.
- It will be convenient to work with the matrix $L = I - \frac{1}{d}A$ instead of the adjacency matrix A.
- The matrix L defined above is called the Normalized Laplacian Matrix of the graph.
- We prove the following basic results of spectral graph theory.

Theorem

Let G be a d-regular undirected graph, and $L = I - \frac{1}{d}A$ be its normalized Laplacian matrix. Let $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be the real eigenvalues of L with multiplicities. Then

1. $\lambda_1 = 0$ *and* $\lambda_n \leq 2$.
2. $\lambda_k = 0$ *if and only if* G *has at least* k *connected components.*
3. $\lambda_n = 2$ *if and only if at least one of the connected components of* G *is bipartite.*

End