Name: ____________________

Entry number: ____________________

There are 2 questions for a total of 10 points.

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1. Answer the following questions:
   (a) (1 point) State true or false: For every matrix $A \in \mathbb{R}^{n \times d}$, there exists a matrix of rank at most k such that $||A - B||_F \leq \frac{||A||_F}{\sqrt{k}}$.

   (a) **False**

   (b) (4 points) Give reason for your answer to part (a).

   Solution: Consider the following counterexample. Let A be a 5×5 unit matrix and $k = 2$. We have $\frac{||A||_F}{\sqrt{k}} = \sqrt{5/2}$. On the other hand, for any matrix with rank at most 2 we have $||A - B||_F \geq ||A - A_2||_F = \sqrt{3}$.

2. Answer the following questions:
   (a) (1 point) State true or false: For every matrix $A \in \mathbb{R}^{n \times d}$, there exists a matrix of rank at most k such that $||A - B||_2 \leq \frac{||A||_F}{\sqrt{k}}$.

   (a) **True**

   (b) (4 points) Give reason for your answer to part (a).

Solution: Let $\sigma_1 \geq \sigma_2 \geq ...$ be the singular values of A. We will use results proved in the class. Let $B = A_k$. Then we have: $||A - B||_2^2 = \sigma_{k+1}^2$. So, we have $k \cdot ||A - B||_2^2 = k \cdot \sigma_{k+1}^2$. This gives the following:

$$\begin{aligned} ||A - B||_2^2 &= \frac{k \cdot \sigma_{k+1}^2}{k} \\ &\leq \frac{\sigma_1^2 + \sigma_2^2 + ... + \sigma_k^2}{k} \\ &\leq \frac{\sum_i \sigma_i^2}{k} = \frac{||A||_F^2}{k} \end{aligned}$$

So, $||A - B||_2 \leq \frac{||A||_F}{\sqrt{k}}$.