There are 9 questions for a total of 60 points.

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1. (5 points) There is a randomised algorithm A for a decision problem X (i.e., answers to instances of the problem are either "yes" or "no") such that for any instance $x \in X$:

$$\mathbf{Pr}[A(x) \text{ is correct}] \geq (1 - \delta)$$

for some constant $0 < \delta < 1/2$. Moreover the running time of A is $t(n)$. Design an algorithm B for the problem X such that:

$$\mathbf{Pr}[B(x) \text{ is correct}] \geq (1 - \delta')$$

where $0 < \delta' < 1/2$ is another constant. Also discuss the running time of the algorithm B.

2. (5 points) Consider an optimisation problem X (assume this is a minimisation problem) and let $OPT(x)$ denote the value of the optimal solution for any instance $x \in X$. Let A be a randomised algorithm such that for any instance $x \in X$:

$$\mathbf{E}[A(x)] \leq c \cdot OPT(x),$$

where $c > 1$ is a fixed constant. Design an algorithm B that with probability at least 0.99 outputs a solution with cost at most $2c$ times the optimal for any input instance. Discuss the running time of the algorithm.

3. (5 points) Given a d-dimensional spherical Gaussian X with $\mathbf{0}$ mean and variance σ in each direction, formulate and prove an appropriate version of the Gaussian Annulus theorem for this case. (*Recall that in the class, we proved for the special case when $\sigma = 1$.*)

4. (5 points) Let x_1 and x_2 be gaussians with means μ_1, μ_2 and variance σ_1, σ_2 respectively. Then show that $y = x_1 + x_2$ is a Gaussian with mean $\mu_1 + \mu_2$ and variance $\sigma_1 + \sigma_2$.

5. (5 points) Recall the discussion on Johnson-Lindenstrauss. We used it as a dimension reduction technique. We showed that there is a randomised mapping $f : \mathbb{R}^d \to \mathbb{R}^k$ with $k < d$ such under this mapping the pairwise distances are preserved with high probability (except for a scaling factor). The mapping is of the form $f(\mathbf{v}) = (\mathbf{u}_1 \cdot \mathbf{v}, ..., \mathbf{u}_k \cdot \mathbf{v})^T$ for randomly chosen vectors $\mathbf{u}_1, ..., \mathbf{u}_k$.

   Show that for any fixed $\mathbf{u}_1, ..., \mathbf{u}_k \in \mathbb{R}^d$ there are vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that distance preserving property does not hold with respect to the mapping f defined with $\mathbf{u}_1, ..., \mathbf{u}_k$.

6. (5 points) Show that if the rows of a square matrix are orthonormal, then the columns are also orthonormal. Is this also true for $d \times n$ matrices where $d < n$? Give reason for your answer.

7. (5 points) Let A be an $n \times d$ matrix ($n \geq d$) such that A has orthogonal columns $\mathbf{v}_1, ..., \mathbf{v}_d$ and lengths $\ell_1 > ... > \ell_d$. What are the right singular vectors and singular values for such a matrix? Give reason for your answer.

8. (10 points) Let $\mathcal{S}^{d-1}$ denote the surface of a unit ball in $\mathbb{R}^d$ (i.e., the set of points $\mathbf{x} \in \mathbb{R}^d$ such that $||\mathbf{x}|| = 1$). Let $0 < \varepsilon < 1/2$. An ε-*covering set* of $\mathcal{S}^{d-1}$ is defined to be any subset $C \subseteq \mathcal{S}^{d-1}$ of points such that for all $\mathbf{x} \in \mathcal{S}^{d-1}$, there exists a point $\mathbf{c} \in C$ such that $||\mathbf{c} - \mathbf{x}|| \leq \varepsilon$. Let $\mathcal{C}$ denote the size of the smallest ε-covering set of $\mathcal{S}^{d-1}$.

   An ε-*packing set* of $\mathcal{S}^{d-1}$ is defined to be any subset $P \subseteq \mathcal{S}^{d-1}$ of points such that for any $\mathbf{x}, \mathbf{y} \in P$, $||\mathbf{x} - \mathbf{y}|| \geq \varepsilon$. Let $\mathcal{P}$ denote the size of the largest packing set of $\mathcal{S}^{d-1}$.

   Give lower and upper bounds for $\mathcal{C}$ and $\mathcal{P}$. Give as tight bounds as you can.

9. (15 points) Consider the following problem of embedding a given undirected graph $G = (V, E)$ over the surface of unit ball in d dimensions (i.e., $\mathcal{S}^{d-1}$ as in the previous problem). This means that for every vertex $v \in V$ you have to give a point x_v in $\mathcal{S}^{d-1}$ such that the following quantity gets maximised:

$$\frac{1}{|E|} \sum_{(u,v) \in E} ||x_u - x_v||^2$$

   Suppose the maximum value achievable for the above quantity (over all possible choices of d) is M. Show that for all $\varepsilon > 0$, there is a set of unit vectors x_v in $\mathbb{R}^{O(\frac{1}{\varepsilon^2} \log 1/\varepsilon)}$ for which the above quantity is at least $(M - \varepsilon)$