COL106: Data Structures and Algorithms

Ragesh Jaiswal, IIT Delhi

- Polynomial-time reduction:
  - Consider two problems X and Y.
  - Suppose there is a *black box* that solves arbitrary instances of problem X.
  - Suppose any arbitrary instance of problem Y can be solved using a polynomial number of standard computational steps and a polynomial number of calls to the black box that solves instance of problem X.
  - If the previous statement is true, then we say that Y is polynomial-time reducible to X. A short notation for this is $Y \leq_p X$.
- Claim 2: Suppose $Y \leq_p X$. If X can be solved in polynomial time, then Y can be solved in polynomial time.
- Claim 3: Suppose $Y \leq_p X$. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

Definition (Independent Set)

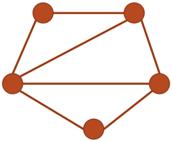
Given a graph $G = (V, E)$, a subset $I \subseteq V$ of vertices is called an independent set of G iff there are no edges between any pair of vertices in I.

Problem

INDEPENDENT-SET: Given a graph $G = (V, E)$ and an integer k, check if there is an independent set of size at least k in G.

Problem

MAXIMUM-INDEPENDENT-SET: Given a graph $G = (V, E)$, output the size of independent set of G of maximum cardinality.



Definition (Independent Set)

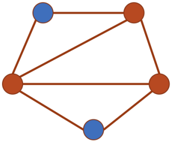
Given a graph $G = (V, E)$, a subset $I \subseteq V$ of vertices is called an independent set of G iff there are no edges between any pair of vertices in I.

Problem

INDEPENDENT-SET: Given a graph $G = (V, E)$ and an integer k, check if there is an independent set of size at least k in G.

Problem

MAXIMUM-INDEPENDENT-SET: Given a graph $G = (V, E)$, output the size of independent set of G of maximum cardinality.



Definition (Independent Set)

Given a graph $G = (V, E)$, a subset $I \subseteq V$ of vertices is called an independent set of G iff there are no edges between any pair of vertices in I.

Problem

INDEPENDENT-SET: Given a graph $G = (V, E)$ and an integer k, check if there is an independent set of size at least k in G.

Problem

MAXIMUM-INDEPENDENT-SET: Given a graph $G = (V, E)$, output the size of independent set of G of maximum cardinality.

- Claim 1: MAXIMUM-INDEPENDENT-SET $\leq_p$ INDEPENDENT-SET.

Definition (Independent Set)

Given a graph $G = (V, E)$, a subset $I \subseteq V$ of vertices is called an independent set of G iff there are no edges between any pair of vertices in I.

Problem

INDEPENDENT-SET: Given a graph $G = (V, E)$ and an integer k, check if there is an independent set of size at least k in G.

Problem

MAXIMUM-INDEPENDENT-SET: Given a graph $G = (V, E)$, output the size of independent set of G of maximum cardinality.

- Claim 1: MAXIMUM-INDEPENDENT-SET $\leq_p$ INDEPENDENT-SET.
- Claim 2: INDEPENDENT-SET $\leq_p$ MAXIMUM-INDEPENDENT-SET.

Definition (Vertex Cover)

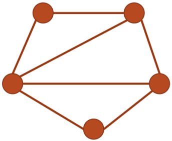
Given a graph $G = (V, E)$, a subset $S \subseteq V$ of vertices is called a vertex cover of G iff for any edge (u, v) in the graph at least one of u, v is in S.

Problem

VERTEX-COVER: Given a graph $G = (V, E)$ and an integer k, check if there is a vertex cover of size at most k in G.

Problem

MINIMUM-VERTEX-COVER: Given a graph $G = (V, E)$, output the size of vertex cover of G of minimum cardinality.



Definition (Vertex Cover)

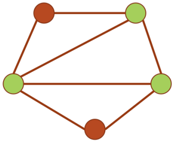
Given a graph $G = (V, E)$, a subset $S \subseteq V$ of vertices is called a vertex cover of G iff for any edge (u, v) in the graph at least one of u, v is in S.

Problem

VERTEX-COVER: Given a graph $G = (V, E)$ and an integer k, check if there is a vertex cover of size at most k in G.

Problem

MINIMUM-VERTEX-COVER: Given a graph $G = (V, E)$, output the size of vertex cover of G of minimum cardinality.



Definition (Vertex Cover)

Given a graph $G = (V, E)$, a subset $S \subseteq V$ of vertices is called a vertex cover of G iff for any edge (u, v) in the graph at least one of u, v is in S.

Problem

VERTEX-COVER: Given a graph $G = (V, E)$ and an integer k, check if there is a vertex cover of size at most k in G.

Problem

MINIMUM-VERTEX-COVER: Given a graph $G = (V, E)$, output the size of vertex cover of G of minimum cardinality.

- Claim 3: MINIMUM-VERTEX-COVER $\leq_p$ VERTEX-COVER.

Definition (Vertex Cover)

Given a graph $G = (V, E)$, a subset $S \subseteq V$ of vertices is called a vertex cover of G iff for any edge (u, v) in the graph at least one of u, v is in S.

Problem

VERTEX-COVER: Given a graph $G = (V, E)$ and an integer k, check if there is a vertex cover of size at most k in G.

Problem

MINIMUM-VERTEX-COVER: Given a graph $G = (V, E)$, output the size of vertex cover of G of minimum cardinality.

- Claim 3: MINIMUM-VERTEX-COVER $\leq_p$ VERTEX-COVER.
- Claim 4: VERTEX-COVER $\leq_p$ MINIMUM-VERTEX-COVER.

- Claim 5: INDEPENDENT-SET $\leq_p$ VERTEX-COVER.

Proof of Claim 5

- Claim 5.1: Let I be an independent set of G, then $V - I$ is a vertex cover of G.

- Claim 5: INDEPENDENT-SET $\leq_p$ VERTEX-COVER.

Proof of Claim 5

- Claim 5.1: Let I be an independent set of G, then $V - I$ is a vertex cover of G.
- Claim 5.2: Let S be a vertex cover of G, then $V - S$ is an independent set of G.

- Claim 5: INDEPENDENT-SET $\leq_p$ VERTEX-COVER.

Proof of Claim 5

- Claim 5.1: Let I be an independent set of G, then $V - I$ is a vertex cover of G.
- Claim 5.2: Let S be a vertex cover of G, then $V - S$ is an independent set of G.
- Claim 5.3: G has an independent set of size at least k if and only if G has a vertex cover of size at most $n - k$.

- Claim 5: INDEPENDENT-SET $\leq_p$ VERTEX-COVER.

Proof of Claim 5

- Claim 5.1: Let I be an independent set of G, then $V - I$ is a vertex cover of G.
- Claim 5.2: Let S be a vertex cover of G, then $V - S$ is an independent set of G.
- Claim 5.3: G has an independent set of size at least k if and only if G has a vertex cover of size at most $n - k$.
- Given an instance (G, k) of the independent set problem, create an instance $(G, n - k)$ of the vertex cover problem, make a single query to the block box for solving the vertex cover problem and return the answer that is returned by the black box. □

- Claim 6: MINIMUM-VERTEX-COVER $\leq_p$ MAXIMUM-INDEPENDENT-SET.

- Claim 6: MINIMUM-VERTEX-COVER $\leq_p$ MAXIMUM-INDEPENDENT-SET.

Proof of Claim 6

- Claim 6.1: G has an independent set of size k if and only if G has a vertex cover of size $n - k$.
- Make a single call to the black box for the maximum independent problem with input G. If the black box returns k, then return $n - k$. □

- Claim 6: MINIMUM-VERTEX-COVER $\leq_p$ MAXIMUM-INDEPENDENT-SET.

Proof of Claim 6

- Claim 6.1: G has an independent set of size k if and only if G has a vertex cover of size $n - k$.
- Make a single call to the black box for the maximum independent problem with input G. If the black box returns k, then return $n - k$. □

Another proof of Claim 6

- MINIMUM-VERTEX-COVER $\leq_p$ VERTEX-COVER
- VERTEX-COVER $\leq_p$ INDEPENDENT-SET
- INDEPENDENT-SET $\leq_p$ MAXIMUM-INDEPENDENT-SET □

Theorem

If $X \leq_p Y$ and $Y \leq_p Z$, then $X \leq_p Z$.

End