COL106: Data Structures and Algorithms

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Network Flow

- Suppose there are four teams in IPL with their current number of wins:
  - Daredevils: 10
  - Sunrisers: 10
  - Lions: 10
  - Supergiants: 8
- There are 7 more games to be played. These are as follows:
  - Supergiants plays all other 3 teams.
  - Daredevils Vs Sunrisers, Sunrisers Vs Lions, Daredevils Vs Lions, Sunrisers Vs Daredevils

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- A team is said to be eliminated if it cannot end with maximum number of wins.
- Can we say that Supergiants have been eliminated give the current scenario?

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  - 4 games between Daredevils and Sunrisers.
- Can we say that Supergiants have been eliminated give the current scenario?

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There are n teams. Each team i has a current number of wins denoted by $w(i)$.There are $G(i,j)$ games yet to be played between team i and j. Design an algorithm to determine whether a given team x has been eliminated.

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- Consider the following flow network

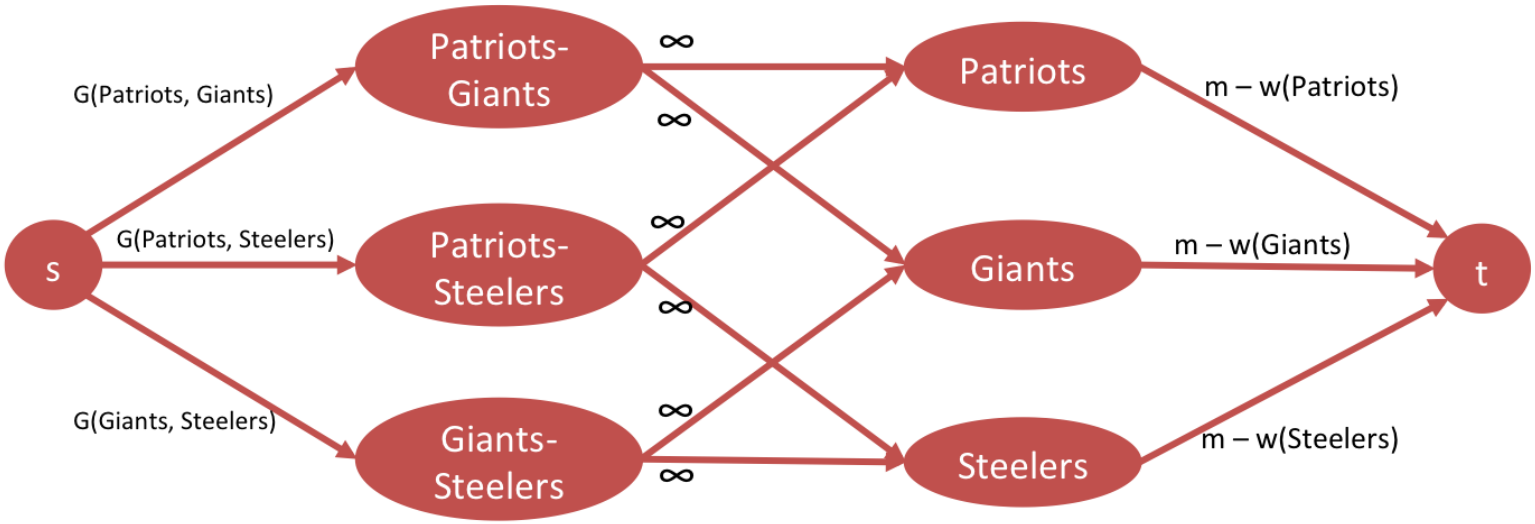

Figure: Team x can end with at most m wins, i.e., $m = w(x) + \sum_j G(x,j)$

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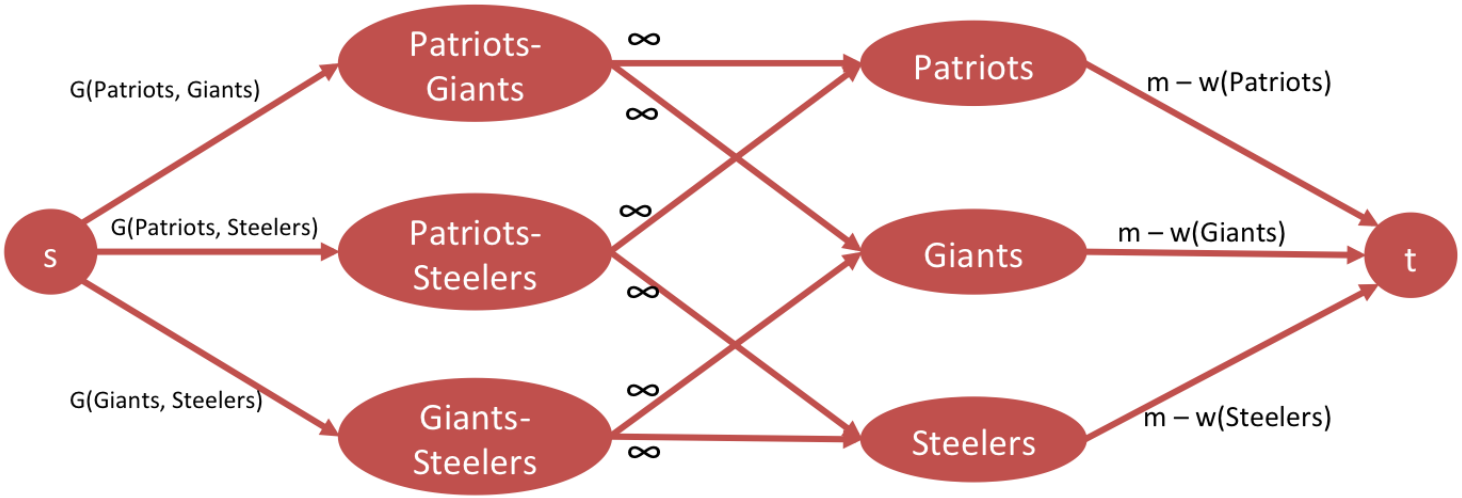
- Claim 1: Team x has been eliminated **iff** the maximum flow in the network is $< g^*$, where $g^* = \sum_{i,j \text{ s.t. } x \notin \{i,j\}} G(i,j)$.



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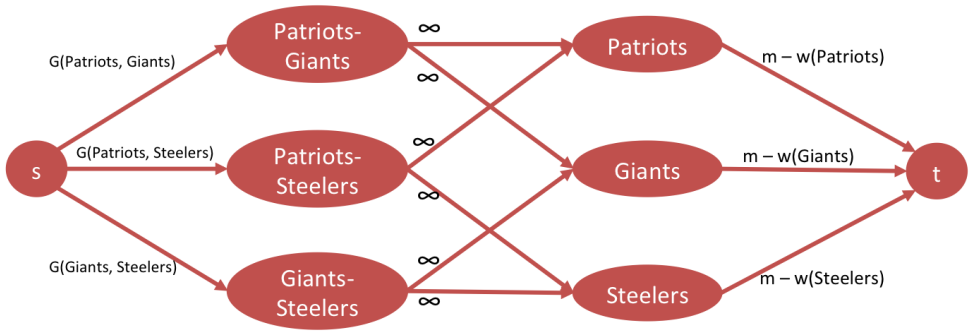

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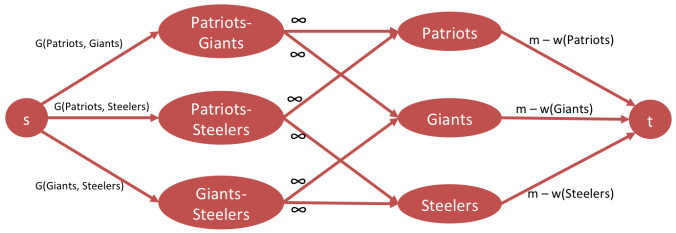
- Can we find such a subset T?



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Proof.

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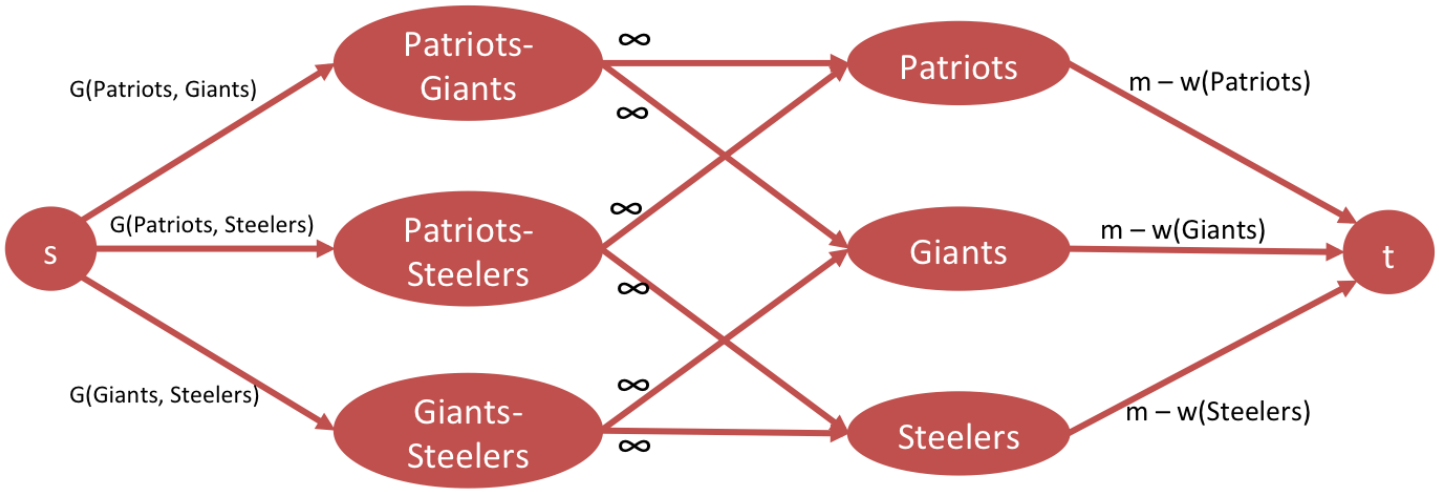
□



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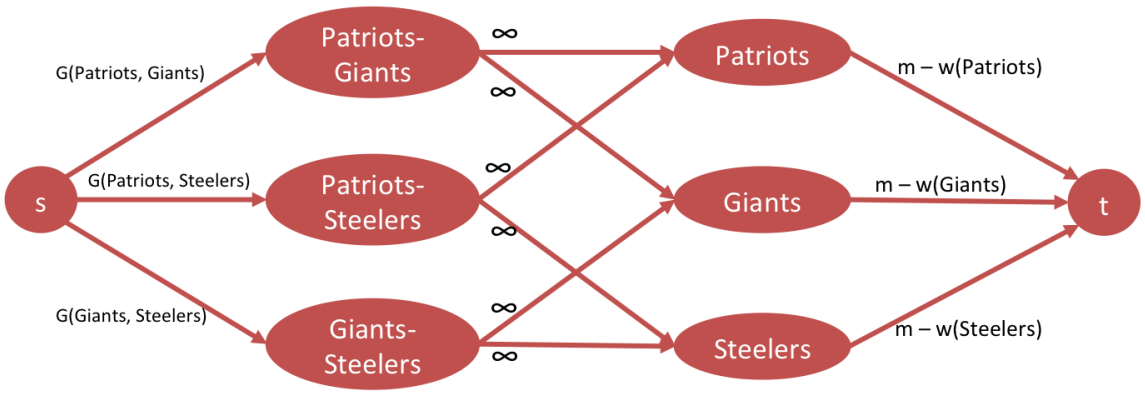
- Claim 1.1: If x has been eliminated, then the max flow in the network is $< g^*$.
- Claim 1.2: If the max flow is $< g^*$, then team x has been eliminated.

Proof of Claim 1.2

- Consider any s-t min-cut (A, B) in the graph.
- Claim 1.2.1: If v_{ij} is in A, then both v_i and v_j are in A.



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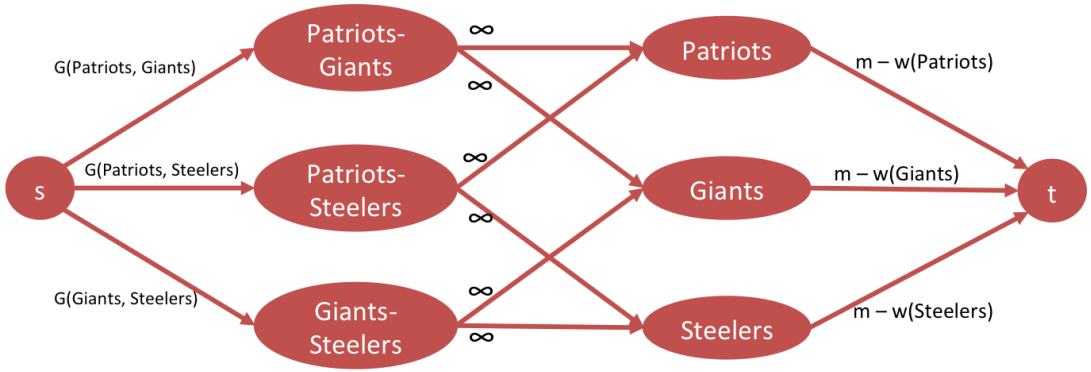

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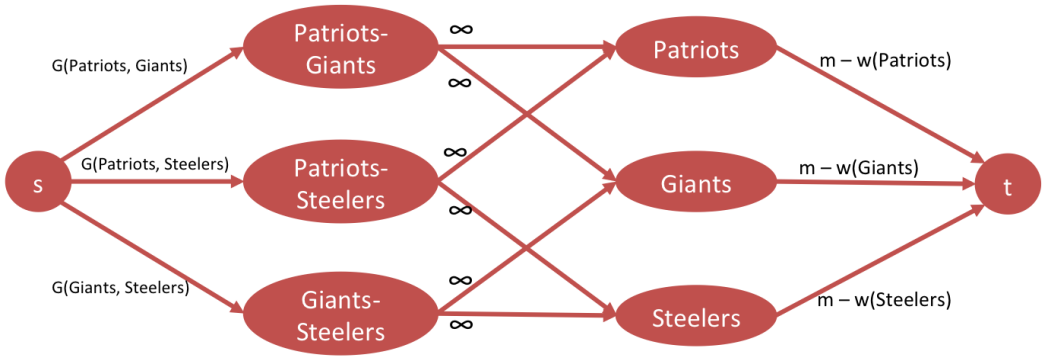

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- Claim 1.2.2: If both v_i and v_j are in A, then v_{ij} is in A.
- Let T be the set of teams such that $i \in T$ **iff** $v_i \in A$. Then we have:

$$C(A,B) = \sum_{i \in T}(m - w(i)) + \sum_{\{i,j\} \not\subset T} G(i,j) < g^*$$

$$\Rightarrow \quad m \cdot |T| - \sum_{i \in T} w(i) + (g^* - \sum_{\{i,j\} \subset T} G(i,j)) < g^*$$

$$\Rightarrow \quad \sum_{i \in T} w(i) + \sum_{\{i,j\} \subset T} G(i,j) > m \cdot |T| \quad \square$$

$\square$

Course Overview

- Basic graph algorithms
- Algorithm Design Techniques:
  - Greedy Algorithms
  - Divide and Conquer
  - Dynamic Programming
  - Network Flow
- Computational Intractability

Computational Intractability

Definition (Efficient Algorithms)

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- Question 1: Given a problem, does there exist an efficient algorithm to solve the problem?
- There are lots of problems arising in various fields for which this question is unresolved.
- Question 2: Are these problems related in some manner?
- Question 3: If someone discovers an efficient algorithm to one of these difficult problems, then does that mean that there are efficient algorithms for other problems? If so, how do we obtain such an algorithm.

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- Polynomial-time reduction:
  - Consider two problems X and Y.
  - Suppose there is a *black box* that solves arbitrary instances of problem X.
  - Suppose any arbitrary instance of problem Y can be solved using a polynomial number of standard computational steps and a polynomial number of calls to the black box that solves instance of problem X.
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- Claim 1: BIPARTITE-MATCHING $\leq_p$ MAX-FLOW.

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- Claim 2: Suppose $Y \leq_p X$. If X can be solved in polynomial time, then Y can be solved in polynomial time.
- Claim 3: Suppose $Y \leq_p X$. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

End