Name: ____________________

Entry number: ____________________

There are 6 questions for a total of 75 points.

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1. Consider functions $f(n) = 10n2^n + 3^n$ and $g(n) = n3^n$. Answer the following:
   (a) (½ point) <u>State true or false</u>: $f(n)$ is $O(g(n))$.

   (a) __________

   (b) (½ point) <u>State true or false</u>: $f(n)$ is $\Omega(g(n))$.

   (b) __________

   (c) (2 points) Give reason for your answer to part (b).

2. (2 points) A rooted tree has a special node r called the *root* and zero or more rooted subtrees $T_1, ..., T_k$ each of whose roots are connected to r. The root of each subtree is called the *child* of r and r is called the *parent* of each child. Nodes with no children are called *leaf* nodes. A *binary tree* is a rooted tree in which each node has at most two children.

   Show by induction that in any binary tree the number of nodes with two children is exactly one less than the number of leaves.

3. (20 points) Given an undirected graph, we call a vertex critical if its removal disconnects the graph. Design an algorithm that finds all the critical vertices in a given graph. Give pseudocode, discuss running time, and give proof of correctness.

4. Consider two vertices s and t in a given graph. A pair of vertices (u, v) (different from s and t) are called bi-critical with respect to s and t if the removal of u and v from the graph disconnects s and t. Suppose we consider graphs where the shortest distance between s and t is strictly greater than $\lceil n/3 \rceil$. With respect to such graphs, answer the questions below:

   (a) (5 points) Prove or disprove the following statement:

   There exists a pair of vertices that are bi-critical with respect to s and t.

(b) (5 points) Design an algorithm for finding a bi-critical pair of vertices with respect to given vertices s and t. Discuss running time. Proof of correctness is not required.

5. (20 points) A directed graph $G = (V, E)$ is called *one-way-connected* if for all pair of vertices u and v there is a path from vertex u to v **or** there is a path from vertex v to u. Note that the **or** in the previous statement is a logical OR and not XOR. Design an algorithm to check if a given graph is one-way-connected. Give pseudocode, discuss running time and give proof of correctness.

6. (20 points) Given a directed acyclic graph $G = (V, E)$ and a vertex u, design an algorithm that outputs all vertices $S \subseteq V$ such that for all $v \in S$, there is an even length simple path from u to v in G.
   (A simple path is a path will all distinct vertices.)

   Give pseudocode, discuss running time, and give proof of correctness.