**CSL851: Algorithmic Graph Theory** **Semester I 2013-2014**

# Lecture 11: September 4

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**Note**: *LaTeX template courtesy of UC Berkeley EECS dept.*



## 11.1 Basic Discrete Probability Theory

Let $\Omega$ be a set and $P(\Omega)$ be its power set. $\Omega$ can be finite or infinite.

**Definition 11.1** *A set* $\Sigma \subseteq P(\Omega)$ *is called a* $\sigma -$ **algebra** *over* $\Omega$ *if*

- $\Omega \in \Sigma$
- $A \in \Sigma \Rightarrow A^c \in \Sigma$
- *For a sequence* $A_1, A_2, \ldots, \in \Sigma$*, we have* $\cup A_i \in \Sigma$

The tuple $(\Omega, \Sigma)$ is called a **measurable-space**. For example:

- $\{\phi, \Omega\}$ is a $\sigma - algebra$
- P($\Omega$) is a $\sigma - algebra$

It also holds: $\bigcap_i A_i \in \Sigma$ if $A_i \in \Sigma$ for all $i$. Note that intersection is over a countable number of sets.

**Definition 11.2** *Let* $(\Omega, \Sigma)$ *be the measurable space. A function* $\mu : \Sigma \rightarrow [0, \infty)$ *is called a* **measure** *if:*

- $\mu(\phi) = 0$
- *For all pairwise disjoint sets* $A_1, A_2, \ldots$ *we have* $\mu(A_1 \cup A_2 \ldots) = \sum \mu(A_i)$

**Definition 11.3** *Let* $P$ *be measure of* $(\Omega, \Sigma)$*.* $P$ *is called a* **probability measure** *if* $P : \Sigma \rightarrow [0, 1]$ *and* $P(\Omega) = 1$*.* $(\Omega, \Sigma, P)$ *is called a* **probability space**.

**Definition 11.4** *A probability space* $(\Omega, \Sigma, P)$ *is called* **discrete** *if* $\Omega$ *is discrete and finite.*
*In a discrete probabilty space* $P$ *the vector* $p = (p(\omega))$ *is called the* **stochastic** *vector,* $p(\omega) = P(\{\omega\})\ \forall \omega \in \Omega$*.*

A **laplacian** probability space $(\Omega, \Sigma, P)$ consists of $\Omega$ finite and $P(\{\omega\}) = 1/|\Omega|\ \forall \omega \in \Omega$. In this case $\Sigma$ is the power set of $\Omega$. A probability measure is also called a **distribution**.

**Proposition 11.5** *Let $\Omega$ be a finite set and $p$ a vector such that $p = (p(\omega))_{\omega\in\Omega}$ and $\sum_{\omega\in\Omega} p(\omega) = 1$ and $p(\omega) \in [0,1]$ then $P(\{\omega\}) = p(\omega)\ \forall\omega \in \Omega$ is a probability measure on $\Omega$.*

**Proof:** For $A \in P(\Omega)$ define $P(A) = \sum_{\omega\in A} p(\omega)$. ■

**Binomial Distribution**: let $n \in \mathbb{N}, 0 < p < 1, \Omega = \{0,1,2,\ldots,n\}, p(\omega) = \binom{n}{\omega}\ p^{\omega}(1-p)^{n-\omega}, \omega \in \Omega$. $P : (p(\omega))_{\omega\in\Omega}$ is a stochastic vector. $B(n,p)$ is the probability measure defined by $P$ and is called the binomial distribution.

For $A \subseteq \Omega$, $B(n,p)(A) = \sum_{\omega\in A} p^{\omega}(1-p)^{n-\omega}$.

**Proposition 11.6** *Let $(\Omega,\Sigma,P)$ be a probability space and $A_1, A_2, \ldots \in \Sigma$ and $B \in \Sigma$. Then,*

- $P(A^c) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- *For $A \subseteq B, P(B/A) = P(B) - P(A)$*
- *For $A \subseteq B, P(A) \leq P(B)$*
- *For $A_1, A_2, \ldots, A_n \in \Sigma$, we have $P(\sum_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$*

**Definition 11.7** *$A \in \Sigma$ is called an* **event**.

**Definition 11.8** *Let $(\Omega,\Sigma,P)$ be a probability space and $A, B \in \Sigma$ with $P(B) > 0$, then $P(A/B) = P(A \cap B)/P(B)$ is the* **conditional probability** *of $A$ assuming the event $B$ or condition on $B$.*

**Definition 11.9** *Let $(\Omega,\Sigma,P)$ be a probability space. Let $A_1, A_2, \ldots, A_n \in \Sigma$.*

- *Let $k \in \{2,\ldots,n\}$. $A_1,\ldots,A_n$ are called $k$-***wise independent***, if for any choice of $k$ sets $B_1, B_2, \ldots, B_k$ from $A_1,\ldots,A_n$, $P(\bigcap_{i=1}^{k} B_i) = \prod_{i=1}^{k} P(B_i)$.*
- *$A_1,\ldots,A_n$ are (mutually)* **independent** *if for all $X \subseteq \{1,2,\ldots,n\}$, we have $P(\bigcap_{i\in X} A_i) = \prod_{i\in X} P(A_i)$.*
  *In particular $P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i)$.*

**Definition 11.10** *Let $(\Omega,\Sigma,P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is called a* **random variable***, if for all open sets $O \subseteq \mathbb{R}, X^{-1}(O) \in \Sigma$.*
*For a finite probability space any $X : \Omega \to \mathbb{R}$ is automatically a random variable because $\Sigma = P(\Omega)$.*

**Notations**: By $P(X \leq x)$, we mean $P(\{\omega|X(\omega) \leq x\})$. Similarly $P(X = x) = P(\{\omega|X(\omega) = x\})$.(X-random variable)

**Definition 11.11** *Let $(\Omega, \Sigma, P)$ be a finite probability space and $X, Y : \Omega \to \mathbb{R}$ be random variables. We can say $X, Y$ are independent if for any choice of $x \in X(\Omega), y \in Y(\Omega)$, we have $P(X = x, Y = y) = P(X = x)P(Y = y)$.*
*Equivalently, $P(X^{-1}(A) \cap Y^{-1}(B)) = P(X^{-1}(A))P(X^{-1}(B))$ $\forall A \subseteq X(\Omega), B \subseteq Y(\Omega)$. This definition is extendable to $n$-random variables.*

**Problems**:
1. Let $(\Omega, \Sigma, P)$ be a probability space, $A, B \in \Sigma$ independent. Show,

- $A, B^c$ are independent
- $A^c, B^c$ are independent

2. Let $(\Omega, \Sigma, P)$ be a laplacian probability space, $X, Y : \Omega \to \mathbb{R}$ be random variables. Given that $X(1) = 2, X(2) = 1, X(3) = 7, Y(1) = 1, Y(2) = 5, Y(3) = 1$, show that $X$ and $Y$ are not independent.