Lecture 9: August 21

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9.1 $2k + 1$ Edge connected graphs

In the previous lecture we had given a procedure to obtain the $2k$ edge connected graphs now lets see the procedure for obtaining $2k + 1$ edge connected graphs which is as follows:

Start with a $2k + 1$ edge connected graph of 2 vertices

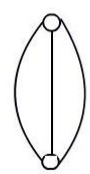

Figure 9.1: For k=1

and repeat any one of the following operations

- Add a new edge.
- Pinch any set of k edges to form a new vertex and add an edge from newly created vertex to an existing vertex.

Claim 9.1 *Every graph obtained by above process is $2k + 1$ edge connected.*

Proof: The proof is similar to the one that has been done in last class for $2k$ edge connected graphs using cuts. ■

Claim 9.2 *Every $2k + 1$ edge connected graph cannot be obtained by above process.*

Proof: It can be seen from the below graph which is a 3 edge connected graph with 4 vertices and cannot be obtained by above process.

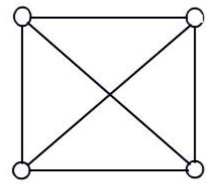

Figure 9.2: 3 Edge connected

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9.2 Lovasz's Splitting Off Theorem

Theorem 9.3 *Given G=(V,E) is a k edge connected graph with k≥2. s∈ V, deg(s) is even and (s,t)∈E, there exists an edge (s,u) ∈E such that between every pair of vertices (u,v)∈* $(V-\{s\})\times(V-\{s\})$ *there are k-edge disjoint paths in* $G^{'} = (V, E\backslash\{(s,t),(s,u)\}\cup(t,u))$.

Proof: The theorem is equivalent to saying that there exists a neighbour of s, u such that after replacing $\{(s,t),(s,u)\}$ with (t,u)(split off operation) every subset of $V-\{s\}$ has cut $\geq k$.

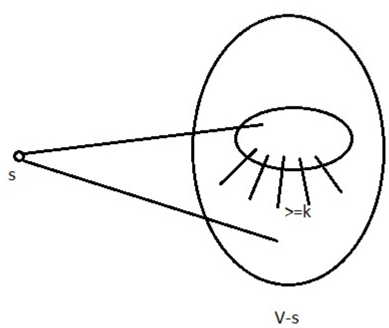

Figure 9.3:

By contradiction if we couldn't split then there exists a subset of $V-\{s\}$ whose cut becomes $< k$ after split off. Observe that subsets which include only either t or u cannot have cut less than k after split off, since the the number of edges across the set remains same(one edge is removed and one edge is added in the split off). So we cannot split off $\{(s,t),(s,u)\}$ if there exists a set $X_u \subset V-\{s\}$ which includes t and u such that $\delta_E(X_u)$ is k or k+1, since number of edges across the set X_u decreases by 2 after split off.

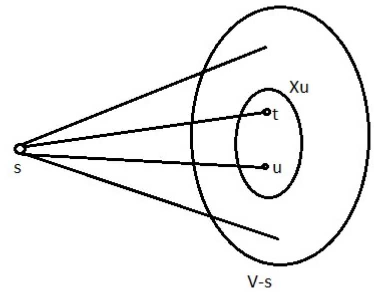

Figure 9.4:

X_u is known as witness set of u, since it is a witness to the fact that $(s,t),(s,u)$ cannot be split and also X_u is strict subset of $V-\{s\}$ because it is actually a cut which should separate any two vertices in $V-\{s\}$. So by our contradiction every neighbour of s has a witness set associated with it.

How many such witness sets are there for given t ?

- case 1: Suppose we have one witness set X that covers all neighbours of s as shown in the figure 9.5.

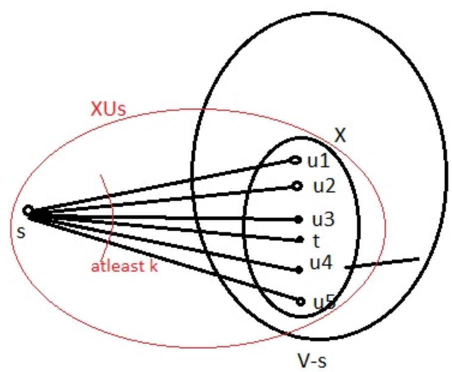

Figure 9.5:

  if we consider the set $XU\{s\}$ it has cut $< k$, since $\delta(X)$ is at most k+1 and $deg(s)$ at least k. Its a contradiction because the original graph is k edge connected.

- case 2: Suppose we have two witness sets A, B that covers all neighbours of s.

  A and B share at least one common vertex t and let $|neighbours(s) \cap A| \geq |neighbours(s) \cap B|$

  if we consider the set $A \cup \{s\}$ then the number of edges across it can be given by

$$\delta(A) - |neighbours(s) \cap A| + |neighbours(s) \cap B| - |neighbours(s) \cap A \cap B|$$

  1. if $|neighbours(s) \cap A \cap B| \geq 2$ then $\delta(A \cup \{s\}) < k$, since $\delta(A)$ is at most $k+1$

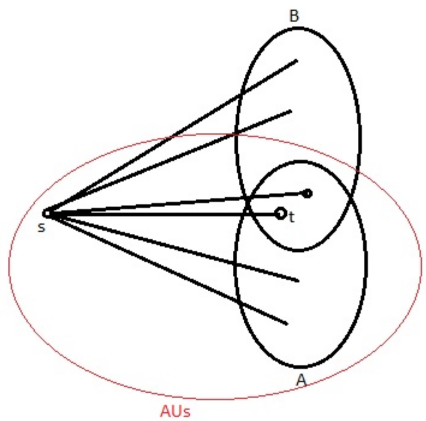

Figure 9.6:

2. if $|neighbours(s) \cap A \cap B| = 1$ then $|neighbours(s) \cap A| > |neighbours(s) \cap B|$ since $deg(s)$ is even and A, B covers all neighbours of s. Hence $\delta(A \cup \{s\}) < k$.

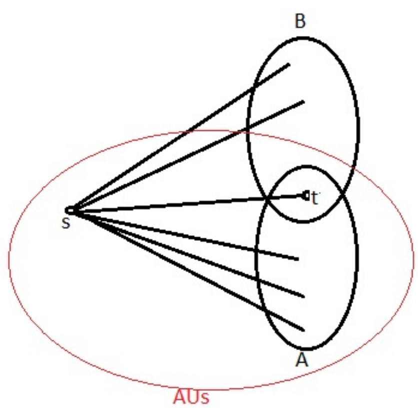

Figure 9.7:

So we formed a set with cut $< k$ which is a contradiction because the original graph is a k edge connected.

- case 3: Suppose we have at least three witness sets A, B, C these will have the following properties

$$t \in A \cap B \cap C \Rightarrow A \cap B \cap C \neq \emptyset$$
$$A - (B \cup C) \neq \emptyset$$
$$B - (A \cup C) \neq \emptyset$$
$$C - (A \cup B) \neq \emptyset$$

since any pair of two sets couldn't cover all the neighbours of s.

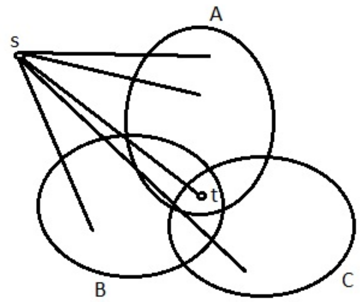

Figure 9.8:

If we consider the edges across the sets A, B, C, $A \cap B \cap C$, $A - (B \cup C)$, $B - (A \cup C)$, $C - (A \cup B)$ we can deduce the following inequality

$$\delta(A) + \delta(B) + \delta(C) \geq \delta(A \cap B \cap C) + \delta(A - (B \cup C)) + \delta(B - (A \cup C)) + \delta(C - (A \cup B)) + 2$$

This inequality can be proved by considering the edges that correspond to cuts of $A \cap B \cap C$, $A - (B \cup C)$, $B - (A \cup C)$, $C - (A \cup B)$ and looking at the number of times each gets counted on both sides of the inequality. each edge is counted at least as many times on the left-hand side as it is counted on the right-hand side, in fact there is at least one edge($s - t$) that is counted three times on the left hand side and once on the right hand side. so the above inequality holds.

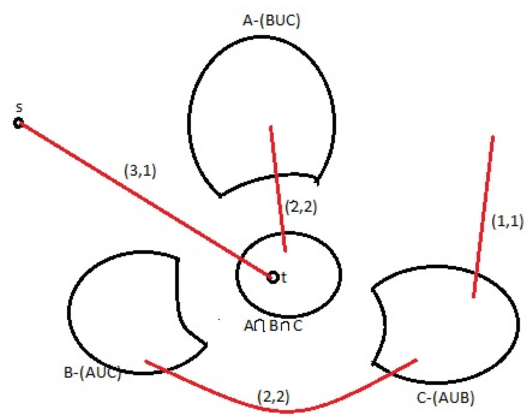

Figure 9.9:

Now observing the fact that A, B, C have cut size at most $k+1$ and $A \cap B \cap C$, $A - (B \cup C)$, $B - (A \cup C)$, $C - (A \cup B)$ have cut size at least k the above inequality implies

$$3(k + 1) \geq 4k + 2 \Rightarrow k \leq 1$$

which is a contradiction since k was assumed to be at least 2.

■

9.3 Edge connectivity augmentation

This is one of the applications of Splitting off theorem

Problem 9.4 *Given a graph G=(V,E) find minimum set of edges $E^{'}$ such that $G^{'} = (V, E \cup E^{'})$ is k-edge connected.*

$Hint$: Suppose we were told that the number of edges incident to v in $E^{'}$. $i.e$ given $deg_{E'}(v)$.

Now create a new node s and add number of edges form every vertex v to s given by $deg_{E'}(v)$. let this graph be $G^{''}$

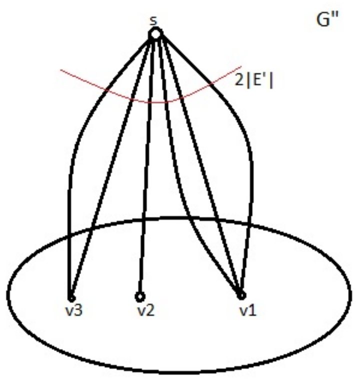

Figure 9.10:

The graph $G^{''}$ is k-edge connected because for every path p between u and v in $G^{'}$ there is a corresponding path $p^{'}$ in $G^{''}$ in which each edge $(a, b) \in E^{'}$ is replaced by distinct pair of edges (a, s) and (s, b) and we were told that adding edges in $E^{'}$ would give a k-edge connected graph. So there are k-edge disjoint paths between every pair of vertices in $G^{''}$.

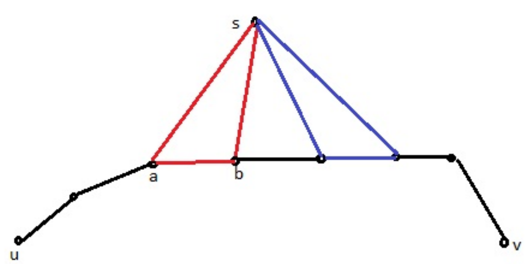

Figure 9.11:

Now if we use splitting off theorem to $G^{''}$ and remove s then the resulting graph is k-edge connected with $|E^{'}|$ additional edges.