CSL851: Algorithmic Graph Theory **Semester I 2013-2014**

Lecture 8: August 19

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This lecture's notes illustrate some uses of various LaTeX macros. Take a look at this and imitate.

8.1 k Arc Connectivity

In the last class we proved the theorem
G is 2 edge connected ⇔ there exists an orientation of G that is strongly connected.
Today we look at k arc connectivity. A graph is k arc connected if k directed disjoint paths between any two vertices u and v exist. Below is an example of a two arc connected graph.

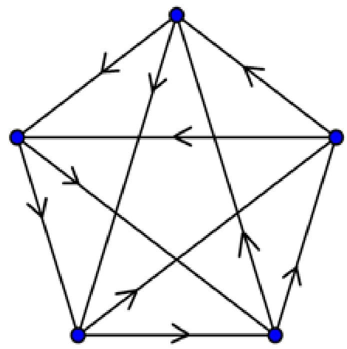

Figure 8.1: *Example of a 2 arc connected graph.*

We prove the following theorem.

Theorem 8.1 *G is 2k edge connected ⇔ there exists an orientation of G that is k arc connected.*

Proof: The proof consists of two parts.

1. G is k arc connected ⇒ G is 2k edge connected.
   The proof of this is via max flow min cut theorem. Assume all edges have capacity one. Consider any S-$\bar{S}$ cut in the graph. Since G is k arc connected there are atleast k edges going outside the cut and atleast k edges coming inside the cut. Now if we remove direction from the edges going across the cut we get atleast 2k edges across any cut in G. The minimum number of edges across any cut is 2k so from min cut max flow theorem we can send atmost 2k units of flow. Hence there are atleast 2k edge disjoint paths across any pair of vertices which implies G is 2k edge connected.

2. G is 2k edge connected ⇒ G is k arc connected.
   For proving this result we need the following theorem:

Theorem 8.2 *Every 2k edge connected graph G can be constructed as follows:*
Start from the multigraph M consisting of two vertices u and v with 2k parallel edges joining u and v. Repeatedly perform one of the following operations:

(a) Add a new edge.

(b) Pinch any set S of k edges. Pinching means to add a new vertex w and to replace each edge uv $\in$ S with the two edges uw and wv.

The proof of this theorem will be given later in the notes. Assuming this theorem is correct we prove by induction on the number of operations performed at every step, that every 2k edge connected graph can be oriented to give a k arc connected graph as follows:

Theorem 8.2 states that any 2k edge connected graph can be constructed using only two steps adding or pinching. Consider a 2k edge connected graph constructed using these steps.
Induction Hypothesis: After any operation a,b mentioned in Theorem 8.2 the graph G remains k arc connected.
Base Case: The initial multigraph M is 2k edge connected by definition. Orient k edges to one direction and rest k edges to opposite direction. This orientation gives a k arc connected graph.
Induction Case: Now suppose we have a k arc connected graph G with orientation D we do one of the following two operations:

- If an edge is added then we can orient it in any direction. The graph is still k arc connected only the number of edges have increased.
- If we pinch an edge set F obtaining a new graph G' then give the pinched edges the orientation they had before pinching. For example an edge uv $\in$ F was directed from u to v before pinching then after pinching we get two new edges uw and wv. Orient them from u to w and from w to v. This yields a new orientation of the graph say D'.

  To prove that G' remains k arc connected after the pinching operation we take any S-$\bar{S}$ cut of the graph. Before the pinching operation $|\delta_{out}(S)| \geq$ k and $|\delta_{in}(S)| \geq$ k since G is k arc connected. After pinching assume the new vertex w $\in$ S then for every edge uv $\in$ F there are new edges uw,wv. These new edges get the same orientation as the edge uv so if uv $\in \delta_{out}(S)$,wv $\in \delta_{out}(S)$ and if uv $\in \delta_{in}(S)$ wv $\in \delta_{in}(S)$. Hence we get $|\delta_{out}(S)| \geq$ k and $|\delta_{in}(S)| \geq$ k. The case when w $\in$ $\bar{S}$ is similar only now we need to consider the new edge uz.

Thus we have proved that we can orient a 2k edge connected graph to get a k arc connected graph.

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8.2 Lovasz's Splitting Off Theorem

Theorem 8.3 *Let $G = (V \cup \{s\}, E)$ be a graph such that degree of s is even and*

$$\forall U; U \subset V \Rightarrow \delta(U) \geq k \tag{8.1}$$

Then there exists (s,u) $\in$ E such that the graph

$$G' = (V \cup \{s\}, E \backslash \{(s,t),(s,u)\} \cup \{(t,u)\})$$

satisfies Condition (1).

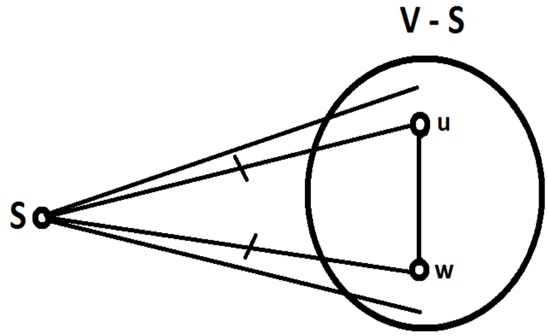

Figure 8.2: *Splitting off edges su and sw to get the edge uw.*

This theorem states that if there is a vertex s of even degree in a k edge connected graph then we can split off s by pairing edges incident on it and replacing them by other edges in the graph and still preserve k edge connectivity. The more general splitting off theorem states that we can simultaneously spit off all edges incident at s. We need this theorem to prove Theorem 8.2. The proof of this theorem would be given in the next class.

8.2.1 Submodularity Property Of Cuts

The submodularity property of cuts states that

Claim 8.4 *For any two sets A and B:*

$|\delta(A)| + |\delta(B)| \geq |\delta(A) \cup \delta(B)| + |\delta(A) \cap \delta(B)|$.

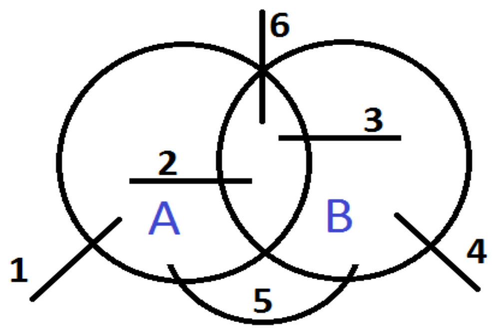

Figure 8.3: *Distinct type of edges between the sets A and B.*

Proof: $|\delta(S)|$ for any set S denotes the number of edges going across the set S. Consider the figure given above. There are 4 distinct sets in the figure A-A∩B, B-A∩B, A∩B, V-A∪B so there will be ${}^4C_2$ types of edges. Edges can be of type 1,2,3,4,5,6. By counting argument we show that the above claim is true. It is very easy to see that all edges of type 1,2,3,4,6 are counted equal number of times on LHS and RHS of the equation. Only edge of type 5 contributes 2 times on LHS and 0 times on RHS. So the above property is proved to be true. ∎

Lemma 8.5 *If we consider two sets A and B of a k edge connected graph G=(V,E) such that A and B are tight cuts and $A \cup B \neq V$ then (A)∪(B) and (A)∩(B) are also tight.*

Proof: A set S is tight if there are exactly k edges going across it. We know that $|\delta(A)| = k$ and $|\delta(B)| = k$ since A and B are tight cuts. On the other hand $|\delta(A) \cup \delta(B)| \geq k$ and $|\delta(A) \cap \delta(B)| \geq k$ since it is a k edge connected graph so each cut has atleast k edges across it. Hence we get

$$|\delta(A)| + |\delta(B)| \leq |\delta(A) \cup \delta(B)| + |\delta(A) \cap \delta(B)|$$

But by the submodularity property of cuts we know that the reverse is true so the only possible case is

$$|\delta(A)| + |\delta(B)| = |\delta(A) \cup \delta(B)| + |\delta(A) \cap \delta(B)|$$

Hence $|\delta(A) \cup \delta(B)| = k$ and $|\delta(A) \cap \delta(B)| = k$ and therefore (A)∪(B) and (A)∩(B)are both tight cuts. ■

Lemma 8.6 *Every minimal k edge connected graph has a vertex of degree k.*

Proof: A minimal k edge connected graph is one where if any edge is deleted from the graph then the graph is no more k edge connected. Let G be a minimal k edge connected graph. Consider the tight cuts in G. Tight cuts are those which have exactly k edges going across them. We first note that every edge has to belong to a tight cut as if it does not belong to a tight cut then the graph is not minimal. Consider minimal tight sets. Minimal tight sets are those which are the smallest tight sets i.e. there are no tight sets within that tight set.

Claim is that there can be no edge within the minimal tight set T. This is because every edge belongs to a tight cut so if there is an edge $e \in T$ then there is a tight cut say S-$\bar{S}$ across which e is going. If $S \cup T \neq V$ then by lemma 8.5 $S \cap T$ and $S \cup T$ are also tight so our set T was not minimal. Hence every minimal tight set is singleton and since it is tight it has k edges across it so we have atleast one vertex of degree k.

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8.2.2 Construction of 2k edge connected graph

Now we prove Theorem 8.2 It is restated here:

Theorem 8.7 *Every 2k edge connected graph can be constructed as follows*
Start from the multigraph M consisting of two vertices u and v with 2k parallel edges joining u and v. Repeatedly perform one of the following operations:

1. *Add a new edge*
2. *Pinch any set S of k edges. Pinching means to add a new vertex w and to replace each edge $uv \in S$ with the two edges uw and wv.*

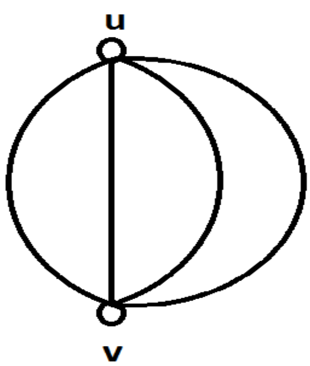

Figure 8.4: Initial multigraph M with k=2

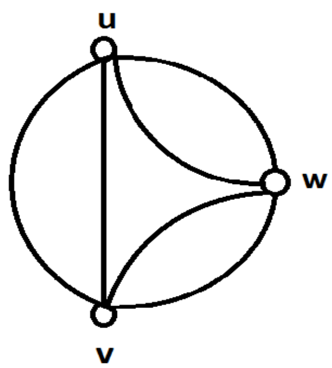

Figure 8.5: Pinching of 2 edges to get the new vertex w

Proof: The proof consists of two parts

1. This process results in a 2k edge connected graph
   The proof is by induction on the number of steps.
   Base Case:Initial multigraph M is 2k edge connected by definition.
   Induction Case: Assume G is 2k edge connected. We do one of the following operations now:

   - If we add an edge then the graph is still 2k edge connected only an additional edge is getting added.
   - We pinch a set F of k edges. Consider any cut of the graph S-$\bar{S}$. Before pinching, atleast 2k edges were going across the cut S-$\bar{S}$. After pinching, we get a new vertex w. Assume the new vertex, w is in $\bar{S}$. Every edge uv $\in$ F was replaced by uw and wv and since w is in $\bar{S}$, k edges are still going across the cut. The other case i.e if w is in S is also similar.

   Hence atleast 2k edges are going across the cut in either case. Thus we see that this process results in a 2k edge connected graph.

2. Any 2k edge connected graph can be built by this process.
   We use Theorem 8.3 to prove this result.
   Assume the more general splitting off theorem to be true. Start with any 2k edge connected graph. If the removal of any edge preserves 2k edge connectivity, remove that edge (reverse of the operation of adding an edge). From lemma 8.6 we know there will be a vertex of degree 2k take that vertex and split off all the 2k edges incident on it (reverse of the operation of pinching k edges). By Lovasz's general splitting off theorem we get a new graph which preserves 2k edge connectivity. We continue this process till we get a multigraph with two vertices and 2k parallel edges. This is our initial multigraph M. Hence we saw how going in the reverse direction we can prove that any 2k edge graph can be built using the construction given in the above theorem .

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