CSL851: Algorithmic Graph Theory **Semester I 2013-2014**

Lecture 7: August 14

Lecturer: Naveen Garg *Scribes: Komal Bansal*

Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. They may be distributed outside this class only with the permission of the Instructor.*

7.1 Ear Decomposition of a Graph

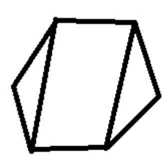
The topic has already been introduced in the previous lecture.

(Quick Recap of what we discussed under this topic is as follows)

- Definition of ear decomposition
- **Lemma**: A graph has an ear decomposition iff it is bridgeless.
- **Lemma**: A two-edge connected graph has following properties:
  - It is bridgeless.
  - Every edge of such a graph is a part of some cycle.
  - Between every pair of vertices of such kind of graph there are atleast two-edge disjoint paths.
- Hence we can say that a graph G has an ear decomposition iff G is two-edge connected.

7.1.1 Open Ear Decomposition

An open ear decomposition is an ear decomposition in which the two endpoints of each ear after the first are distinct from each other.



Open Ear
Decomposition

7.1.2 Odd Ear Decomposition

An ear decomposition is odd if each of its ears uses an odd number of edges.

Lemma 7.1 *A graph G=(V,E) has an odd ear decomposition iff G is factor-critical.*

Proof: Let us first prove the easier part "*If G has an odd ear decomposition, then it is factor critical*".

Using induction.
Base Case: Consider a graph made up of an odd length cycle. This graph has an odd ear decomposition trivially. Now if we keep any vertex unmatched, we will be left with even number of vertices, which can be matched among themselves.

Now let us augment the graph with an odd ear. since this is an odd ear, number of vertices on this path would be even. There occur two cases now:

- Take one vertex from cycle and keep it unmatched. Since we have even vertices in the cycle, they can be matched among themselves. Also odd path will consist of even vertices. Hence they can be matched among themselves as shown in figure 1.

- Now take one vertex from odd path. Now we have odd number of vertices on this odd path. So, we use one vertex from cycle to match its vertices completely. And we have already shown that even if one vertex is taken out from cycle, others can be matched among themselves as shown in figure 2. Hence matching can be done.

  Hence this statement is true for base case.

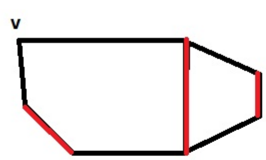


figure 1

Keep v unmatched

Matching edges

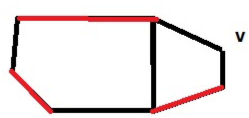


figure 2

Induction Hypothesis: Let us assume that the above lemma holds for a graph made up of k odd ears. Now we will prove that it holds for a graph made up of k+1 odd ears.

If we take a vertex out of the graph made up of k ears, we can match all its remaining vertices (by induction hypothesis). And the (k+1)th odd ear is left with even number of vertices as unmatched, so they can be matched among themselves.[figure 3]

Now take a vertex from (k+1)th ear. Since we will be left with odd number of vertices in this ear, we need one vertex from the graph with k ears. Again by induction hypothesis, if one vertex is taken from this graph with k ears, rest of the vertices can be matched.[figure 4]

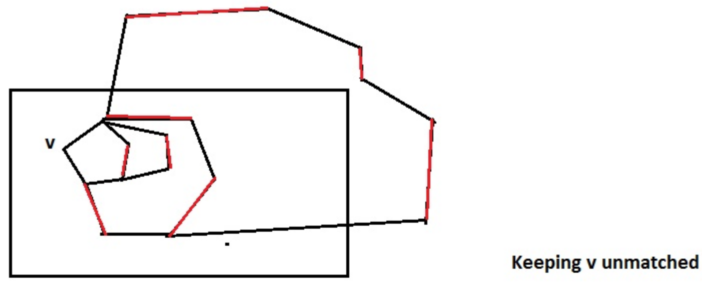


figure 3

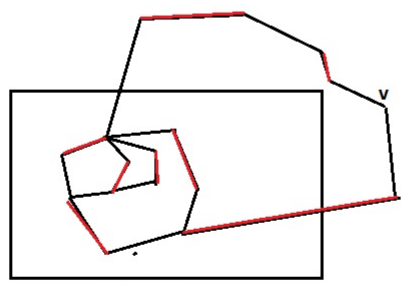


figure 4

Hence one direction is proved.

Now Let us prove the other part "*If G is factor critical, then it has an odd ear decomposition*".
Again using proof by induction.

Base Case: Consider an odd cycle. Before using this we need to prove that *an odd cycle must exist in a factor critical graph.*

Using proof by contradiction: Let G=(V,E) be a factor critical graph and it consists of only even cycle, even and odd paths. Then, we can divide them in the form of bipartite graph. and either both sides will have equal number of vertices or one would have smaller number of vertices than the other. Now, in both the cases we can always take one vertex out such that one of the set in bipartite distribution, will have larger number of vertices. So, we won't be able to match all of them. Hence the graph won't be factor critical.
Hence contradiction.

Therefore an odd cycle must exist in a factor critical graph.

Let us define two matchings as follows:
M1 = {e | e in E, all vertices in {V-r} are matched }
M2 = {e | e in E, all vertices in {V-b} are matched }

Now, if we take symmetric difference of M1 and M2 (i.e. the edges which are either in M1 or in M2 but not in both are taken) we will get an even length alternating path as shown in figure 5.

If we put the egde (r,b) on this path, we will get an odd cycle. And we can take this cycle as an odd ear. Hence odd ear decomposition is possible.

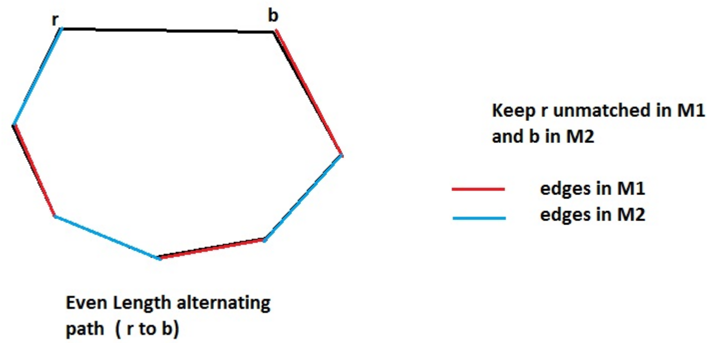


figure 5

Induction Hypothesis: Let S be vertex set covered by odd ear decomposition.

Now we need to prove that if we augment S, then also odd ear decomposition can be done.

Note: We will maintain an invariant here, that if set S is already covered by odd ear decomposition, then no edge of matching M1 will cross S. This can easily be done by keep v1 fixed throughout the proof.

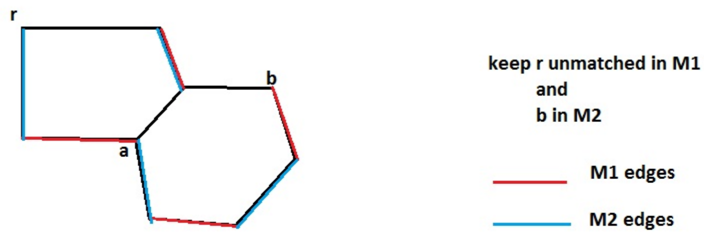
Since, the graph is connected, an edge (a,b) must be there such that a $\in$ S and b $\notin$ S.

Suppose V is set of all the vertices.

Let us redefine two matchings as follows:
M1= {e | e $\in$ E, all vertices in V-r are matched }
M2= {e | e $\in$ E, all vertices in V-b are matched }

Now by the construction of M2 we know that no edge of M1 is going across S. But in M2 matching we have kept b as unmatched, so there must exist some vertex in S let say s, which must be matched across S in M2.

Again if we take symmetric sum of M1 and M2, we will get an even length alternating path starting at s and ending at b as shown in figure. Now if we add (a,b) to this we will get an odd length path, which can be taken as an odd ear. Also, the set S can be decomposed into odd ears (given by induction hypothesis). Hence, in this manner we can augment S and prove that all the factor critical graphs can be decomposed into odd ears.[figure 6]



Even length alternating path from b to a

figure 6

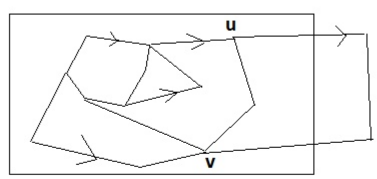
Hence Proved. ■

7.2 Graph Orientations

An orientation of an undirected graph is an assignment of a direction to each edge, turning the initial graph into a directed graph, such that it satisfies some required property.

e.g. Given a 2-egde-connected graph, orient its edges so that it becomes strongly connected

As we have already discussed in the previous lecture that every 2-edge-connected graph has an ear decomposition. So, we will decompose the given graph into ears and assign some direction to each ear such that all the edges in that ear are pointing in the same direction.



Assigning direction to ears

figure 7

This construction will give us a strongly connected graph.

Proof: Using induction

Consider an ear in the form of simple cycle only. If we assign some direction to this cycle any pair of vertices belonging to this cycle are reachable from each other.

Now if we add some ear to this cycle and assign any direction to that then also it remains strongly connected. Let us denote the end-points of this ear as 'u' and 'v'. Let us assign direction as u to v. Now, since u and v are parts of cycle, there must exist a path from v to u. Hence again we get another cycle (u,v,u). Therefore, every pair of vertices beloging to this are again reachable from each other. Same can be claimed for further addition of ears.

Hence graph becomes strongly connected. ∎

Lemma 7.2 *Every strongly connected graph is two-edge connected in undirected form.*

Proof: Every strongly connected graph satisfies following conditions.

$$| \delta^{in}(\mathrm{S}) | \geq 1$$
$$| \delta^{out}(\mathrm{S}) | \geq 1$$
$$\forall \ \mathrm{S} \subset \mathrm{V}$$

For undirected version above kind of graph will satisfy following condition.

$$| \delta_G(\mathrm{S}) | \geq 2$$

Hence the graph will be two-edge connected.

Also, every pair of vertices (u,v) in G has a path from u to v and v to u, therefore, clearly in undirected version we have two edge-disjoint paths from u to v and v to u. ∎

Lemma 7.3 *Every edge of a strongly connected graph is a part of a cycle.*

Proof: We just proved that every strongly connected graph is two-edge connected. And in the previous lecture we proved that every edge of a two-edge connected graph is a part of a cycle. Hence we can say that every edge of a strongly connected graph is also a part of a cycle.
Hence Proved. ∎